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Periodic automorphisms of the hyperfinite factor of type II_1

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Introduction

There are many constructions of factors which give rise to the hyperfinite factor of type II_1, that we shall throughout denote by R. For instance 1) any infinite tensor product of a countable number of matrix algebras with respect to their traces, 2) the group measure space construction from an ergodic measure preserving transformation, 3) the left regular representation of a locally finite discrete group with infinite conjugacy classes.

To each of those ways of obtaining R correspond automorphisms of R. Two automorphisms α and β of R are conjugate when for some automorphism σ of R one has $\sigma\alpha\sigma^{-1}=\beta$.

The simplest nontrivial problem of noncommutative ergodic theory is certainly the problem of classifying, up to conjugacy, the periodic automorphisms of R.

It turns out that a complete classification is possible, by means of very simple invariants that we shall now describe.

We note first that the problem of conjugacy splits into two problems:

a) The problem of outer conjugacy: decide when given $\alpha, \beta\in \mathrm{Aut}\, R$ there exists an inner automorphism $\mathrm{Ad}\, W$ such that β is conjugate to $\mathrm{Ad}\, W\cdot\alpha$.

b) The problem of inner conjugacy: for $\alpha\in\mathrm{Aut}\, R$ decide which W, unitaries in R, are such that $\mathrm{Ad}\, W\cdot\alpha$ is conjugate to α.

For solving problem a) we first define two invariants of outer conjugacy:

1. $p_0(\alpha)$ is the outer period of α defined as the integer such that, for $n\in\mathbf{Z}$, $\alpha^n\in\mathrm{Int}\, R\Leftrightarrow n\in p_0(\alpha)\mathbf{Z}$.

2. $\gamma(\alpha)$ is a complex number of modulus 1 defined by the implication: U unitary in R, $\alpha^{p_0(\alpha)}=\mathrm{Ad}\, U \Rightarrow \alpha(U)=\gamma U$. One checks by direct computation (prop. 1.4) that p_0 and γ are invariants of outer conjugacy classes and that $\gamma(\alpha)^{p_0(\alpha)}=1$.

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We exhibit for each couple $p\in\mathbf{N}$, $\gamma\in\mathbf{C}$, with $\gamma^p=1$, an automorphism of R, s_p^γ, of period equal to $p\cdot$Order γ and such that

$$p_0(s_p^\gamma) = p, \quad \gamma(s_p^\gamma) = \gamma \quad \text{(prop. 1.6)}.$$

We prove that the invariants p_0, γ completely classify the periodic automorphisms of R, up to outer conjugacy, so that any periodic automorphism of R is outer conjugate to one (and only one) of the s_p^γ (thm. 6.2).

The proof relies on the introduction of a group structure on the set Br_p of outer conjugacy classes of automorphisms with outer period p. One checks that if α and β are such classes then $\alpha\otimes\beta$ is also a class belonging to Br_p, as well as the class of the opposite α^0 of α, once $R\otimes R$ and R^0 (the opposite factor of R) are identified with R by some isomorphism (the classes $\alpha\otimes\beta$ and α^0 being of course independent of this isomorphism).

Once this is done one proves that Br_p is a group with inverse operation $\alpha\to\alpha^0$ and that γ is an isomorphism of Br_p onto the group of pth roots of 1 in $\mathbf{C}$.

The proof of the injectivity of γ, i.e., of the uniqueness of the outer conjugacy class with outer invariants $(p, 1)$, is obtained thanks to the technique of central sequences, as used by D. McDuff in [7] (see thm. 5.1).

The reader who is familiar with the construction of the Brauer group $B(k)$ of an arbitrary commutative field k will recognise the analogy with the construction of Br_p above — the objects that we study are periodic automorphisms of R, to the concept of similarity of simple central algebras over k corresponds the concept of outer conjugacy of two periodic automorphisms. The role of division algebras is played by the *minimal* periodic automorphism: α is called minimal periodic when its period is the smallest period of its outer conjugacy class. Exactly as any central simple algebra over k is the tensor product of a unique division algebra by a matrix algebra $M_n(k)$, we have that any periodic automorphism of R is the tensor product of a minimal periodic automorphism (uniquely determined up to conjugacy) by an inner automorphism — (thm. 1.11). Moreover the minimal automorphisms are also characterised by their fixed point algebra being a factor (thm. 2.5).

For each $p\in\mathbf{N}$, $\gamma\in\mathbf{C}$, $\gamma^p=1$, the automorphism s_p^γ is the unique minimal automorphism of the outer conjugacy class with outer invariants p, γ.

Also, the tensor product of two division algebras over k can fail to be a division algebra and in the same way the tensor product of two minimal automorphisms can fail to be minimal. The answer to problem b) is obtained by defining the inner invariant $\varepsilon(\alpha)$ of an arbitrary periodic automorphism α of R as the spectral measure (defined only up to rotation) corresponding to the trace vector and an arbitrary $U\in R$ such that $\alpha^{p_m(\alpha)}=\operatorname{Ad} U$, where $p_m(\alpha)$ is the minimal period of α. It turns out that p_0, γ and ε form a complete system of invariants for periodic automorphisms of R, the only relations being that $\gamma^{p_0}=1$ and that the support of ε lies in the nth roots of 1 for some n (thm. 1.11).

This also allows to solve the problem of weak equivalence:

c) For α and $\beta\in\operatorname{Aut} R$, when is there a $\sigma\in\operatorname{Aut} R$ such that $\sigma[\alpha]\sigma^{-1}=[\beta]$? (where the full group $[\alpha]$ of α is defined in classical terms (see for instance [4] def. 1.5.4)). The invariants of weak equivalence are $p_0(\alpha)$, Order $\gamma(\alpha)=c(\alpha)$ and the symbols of Legendre $(j/p)=\pm 1$, where p is a prime dividing $c(\alpha)$, and $\gamma(\alpha)=\exp i2\pi j/c$, with two additional symbols $\varepsilon(j), \omega(j)$ (resp. one: $\varepsilon(j)$) when $c(\alpha)$ is divisible by 4 (resp. 2 but not 4) which are classical in elementary arithmetic.

It turns out that they are complete invariants of weak equivalence the only relation being that $c(\alpha)$ divides $p(\alpha)$ (thm. 6.5).

We then apply these results to simple questions of noncommutative ergodic theory and we get the following answers: A periodic α is conjugate to the opposite of its inverse if and only if $\gamma(\alpha)^2=1$ (thm. 7.2).

A periodic α is an infinite tensor product of inner automorphisms if and only if $\gamma(\alpha)=1$ (thm. 7.9).

Also we determine conditions, of an arithmetical nature, under which α is an infinite tensor product of automorphisms of finite dimensional factors (thm. 7.4 (c)).

Then we prove that any periodic automorphism α of R admits very good approximation by finite dimensional automorphisms, in the sense that $\alpha(P_n)=P_n$, $\forall n\in\mathbf{N}$ for some increasing sequence of finite dimensional subalgebras of R with $\bigcup_{n=1}^{\infty} P_n$ dense in R.

This, of course, implies that the cross products or fixed point von Neumann algebras of arbitrary periodic automorphisms of R are hyperfinite (see remark 7.10 on this point).

Finally we give an example of a (periodic) automorphism of R which has no square root, and give the conditions (thm. 7.7) (namely $c(=\text{Order }\gamma)$ odd) under which s_p^γ has a square root.

I. Construction of the automorphisms $s_p^\gamma, p\in\mathbf{N}, \gamma^p=1$

Let N be a factor, $\alpha\in\operatorname{Aut} N$, then we define two numbers, $p_0(\alpha)$ and $\gamma(\alpha)$ as follows: [1])

$$\{n\in\mathbf{Z}, \alpha^n\in\operatorname{Int} N\} = p_0(\alpha)\mathbf{Z} \quad\text{and}\quad p_0(\alpha)\in\mathbf{N}, \tag{1.1}$$

$$(\alpha^{p_0(\alpha)} = \operatorname{Ad} U,\ U \ \text{ unitary in } \ N) \Rightarrow \alpha(U) = \gamma(\alpha)U.$$

We see that for each α, $p_0(\alpha)$ is an integer, that we call the outer period of α; it is 0 if all the nonzero powers of α are outer.

[1]) See remark 6.8 for a cohomological interpretation of $\gamma(\alpha)$ as an obstruction.

Also we see that $\gamma(\alpha)$ is a complex number of modulus 1, independent of the choice of U such that $\alpha^{p_0(\alpha)} = \mathrm{Ad}\, U$, and satisfying

$$\gamma(\alpha)^{p_0(\alpha)} = 1 \tag{1.2}$$

because $\alpha^{p_0(\alpha)}(U) = \gamma(\alpha)^{p_0(\alpha)} U$ and $\alpha^{p_0(\alpha)}(U) = UUU^* = U$.

Definition 1.3. α and $\beta \in \mathrm{Aut}\, N$ are called *outer conjugate* iff there exists a $\sigma \in \mathrm{Aut}\, N$ such that β and $\sigma\alpha\sigma^{-1}$ have the same image in $\mathrm{Out}\, N = \mathrm{Aut}\, N / \mathrm{Int}\, N$.

For W unitary in N, put for $\alpha \in \mathrm{Aut}\, N$, ${}_W\alpha = \mathrm{Ad}\, W \cdot \alpha$. When W varies, the ${}_W\alpha$ form the class of α in $\mathrm{Out}\, N$ hence the $\beta \in \mathrm{Aut}\, N$ which are outer conjugate to α are all automorphisms conjugate to some ${}_W\alpha$, W unitary in N.

Proposition 1.4. *If α and β are outer conjugate then $p_0(\alpha) = p_0(\beta)$, $\gamma(\alpha) = \gamma(\beta)$; $(p_0(\alpha), \gamma(\alpha))$ is called the outer invariant of α.*

Proof. The first equality is clear. To prove the second we can assume that $\beta = {}_W\alpha$ for some W unitary in N. Then let $p = p_0(\alpha)$, $\gamma = \gamma(\alpha)$; $\alpha^p = \mathrm{Ad}\, U$, $\alpha(U) = \gamma U$, then we have

$$({}_W\alpha)^p = \mathrm{Ad}\,\big(W\alpha(W)\ldots\alpha^{p-1}(W)\,U\big),$$

$${}_W\alpha\big(W\alpha(W)\ldots\alpha^{p-1}(W)\,U\big) = W\alpha(W)\ldots\alpha^{p-1}(W)\,UWU^*\alpha(U)\,W^*,$$

hence ${}_W\alpha\big(W\alpha(W)\ldots\alpha^{p-1}(W)\,U\big) = W\alpha(W)\ldots\alpha^{p-1}(W)\,U\gamma$ Q.E.D

We shall now fix our notations, as far as the simple classification of *inner* periodic automorphisms is concerned, for the case of factor N of type II_1 with canonical trace τ $(\tau(1)=1)$.

Let $\alpha = \mathrm{Ad}\, U$ be periodic, then the unitary U which is uniquely determined by α up to multiplication by a $\lambda \in \mathbf{C}$, $|\lambda| = 1$, has the property that U^p is a scalar λ_0 for $p =$ period α. It follows that U is a finite linear combination of its spectral projections corresponding to the pth roots a_j of λ_0, say $U = \sum_{j=1}^{p} a_j e_j$, where e_j is the spectral projection of U corresponding to $\{a_j\}$.

We define now the inner invariant $\varepsilon(\alpha)$ to be the probability measure $\Sigma\tau(e_j)\varepsilon_{a_j}$, determined up to a rotation on $\mathbf{T} = \{z \in \mathbf{C}, |z| = 1\}$. It is easy to see that two inner automorphisms α and β are conjugate iff $\varepsilon(\alpha) = \varepsilon(\beta)$ and that all probability measures on $\mathbf{T}$ which have support contained in the pth roots of some $\lambda_0 \in \mathbf{T}$ arise as $\varepsilon(\alpha)$. For $\alpha \in \mathrm{Aut}\, N$, α periodic with outer invariants p_0, γ we put $p_m = p_0 \cdot$ Order γ and we put $\varepsilon(\alpha) = \varepsilon(\alpha^{p_m})$. (Check that $\alpha^{p_m} \in \mathrm{Int}\, N$.)

Theorem 1.5. *Let N be the II_1 hyperfinite factor R. Two periodic automorphisms $\alpha, \beta \in \mathrm{Aut}\, R$ are conjugate if and only if they have the same outer and inner invariants* (i.e. $p_0(\alpha)$, $\gamma(\alpha)$ *and* $\varepsilon(\alpha)$).

This paper is entirely devoted to prove theorem 1.5. We spend the rest of this paragraph in giving a simple description of automorphisms, the $s_p^\gamma \otimes \mathrm{Ad}\, V$, having prescribed outer and inner invariants. Our first task is to describe the automorphisms $s_p^\gamma, p\in \mathbf{N}, \gamma^p=1$ which have outer invariant (p, γ) and trivial inner invariant. For $p=1$ we let s_1^1 be the identity automorphism of R. Let $p\neq 1$. Then we write R as the infinite tensor product, indexed by $\mathbf{N}$, of the couples (F_p, Canonical trace on F_p) where F_p is the $p\times p$ matrix algebra over $\mathbf{C}$ with matrix units $(e_{i,j})_{i,j=1\dots p}$.

For $q\in\mathbf{N}$, let π_q be the canonical isomorphism of F_p onto a subfactor F_p^q of R, such that $\pi_q(x)=1\otimes\cdots\otimes 1\otimes x\otimes 1\cdots$.

Let $e_{ij}^q=\pi_q(e_{ij})$ and θ be the shift: $\theta\pi_q(x)=\pi_{q+1}(x), x\in F_p$. The shift θ is an isomorphism of R onto the commutant of F_p^1 in R.

Let $\gamma\in\mathbf{C}, \gamma^p=1$, and $U_\gamma\in F_p^1$ be the unitary:

$$U_\gamma = \sum_{j=1}^{p} \gamma^j e_{jj}^1.$$

We define a unitary $v_\gamma\in(F_p^1\cup F_p^2)''$ [2]) by the formula:

$$v_\gamma = e_{p1}^1\theta(U_\gamma^*)+\sum_{j=1}^{p-1} e_{j,j+1}^1.$$

The following proposition is at the same time the definition of the automorphism s_p^γ of R.

Proposition 1.6. *Let p and γ be as above.*

(a) *The sequence of inner automorphisms of R defined by*

$$\alpha_n = \mathrm{Ad}\left(v_\gamma\theta(v_\gamma)\theta^2(v_\gamma)\theta^3(v_\gamma)\cdots\theta^n(v_\gamma)\right)$$

converges pointwise strongly to an automorphism s_p^γ of R.

(b) *The pth power $(s_p^\gamma)^p$ of this automorphism is equal to* $\mathrm{Ad}\, U_\gamma$ *and* $s_p^\gamma(U_\gamma)=\gamma U_\gamma$.

(c) *The outer invariant of s_p^γ is equal to (p,γ), its inner invariant is $\{\varepsilon_1\}$.*

Proof. (a) Let m be given and $x\in F_p^{(1,m)}=(\bigcup_1^m F_p^q)''$. Then for $n\geq m$ we have $[\theta^n(v),x]=0$ for any $v\in R$. It follows that $\alpha_n(x)=\mathrm{Ad}\left(v_\gamma\theta(v_\gamma)\cdots\theta^{m-1}(v_\gamma)\right)(x)=\alpha_{m-1}(x)$ so that the sequence $(\alpha_n(x))_{n\in\mathbf{N}}$ is constant for $n\geq m$. For each n, α_n is an isometry in the L^2 norm of R; it follows hence from the strong density in R of the subalgebra $\bigcup_{m=1}^\infty F_p^{(1,m)}$ that there exists an homomorphism s_p^γ of R into R such that

$$s_p^\gamma(x) = \lim_{n\to\infty}\alpha_n(x), \quad x\in R.$$

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[2]) From now we let $F_p^{(i,j)}=(\sum_{i\leq q\leq j} F_p^q)''$, hence for instance $v_\gamma\in F_p^{(1,2)}$.

We shall now prove that $(s_p^\gamma)^p = \mathrm{Ad}\, U_\gamma$. It will hence follow that s_p^γ is surjective and is an automorphism of R.

(b) We first have, using the equality $s_p^\gamma(U_\gamma) = \alpha_0(U_\gamma)$, that:

$$s_p^\gamma(U_\gamma) = \mathrm{Ad}\, v_\gamma(U_\gamma) = \mathrm{Ad}\left(\sum_{j=1}^{p} e_{j,j+1}\right)(U_\gamma) = \left(\sum_1^p e_{j,j+1}\right)\left(\sum_1^p \gamma^k e_{kk}\right) \cdot$$

$$\left(\sum_1^p e_{l+1,l}\right) = \sum_1^p \gamma^{j+1} e_{jj} = \gamma U_\gamma. \text{ [3]}$$

We end the proof of (b) by showing by induction on m that the following statement is true:

(1.7) $\qquad \forall \gamma \in \mathbf{C}, \gamma^p = 1, \quad x \in F_p^{(1,m)}$ one has $(s_p^\gamma)^p(x) = U_\gamma x U_\gamma^*$.

We assume that the statement is true for m, we prove it for $m+1$, its truth for $m=1$ also follows from this computation.

Put, for $x \in R$, $\beta(x) = \lim_{n\to\infty} \mathrm{Ad}\,(\theta(v_\gamma)\theta^2(v_\gamma)\cdots\theta^n(v_\gamma))\,(x)$ then, as above, β is an homomorphism of R into R, which leaves F_p^1 pointwise invariant and satisfies the equality:

(1.8) $$\beta(\theta(x)) = \theta(s_p^\gamma(x)), \quad x \in R.$$

Take $x \in F_p^{(1,m+1)}$, $x = \sum_{i,j} e_{ij}^1 \theta(x_{ij})$ with $x_{ij} \in F_p^{(1,m)}$. From (1.8) and the induction hypothesis we conclude that:

$$\beta^p \theta(x_{ij}) = \theta((s_p^\gamma)^p(x_{ij})) = \theta(U_\gamma)\theta(x_{ij})\theta(U_\gamma)^* \quad \text{for} \quad i, j = 1, \dots, p;$$

and hence, using the equalities $\beta^p(e_{ij}^1) = e_{ij}^1$, for $i, j = 1, \dots, p$:

(1.9) $$\beta^p(x) = \theta(U_\gamma) x \theta(U_\gamma)^*.$$

But we have $s_p^\gamma = \mathrm{Ad}\, v_\gamma \cdot \beta$, hence (1.7) will follow from

(1.10) $$v_\gamma \beta(v_\gamma) \dots \beta^{p-1}(v_\gamma) = U_\gamma \theta(U_\gamma^*).$$

To prove (1.10) we just have to use the equality $\beta\theta(U_\gamma) = \theta(s_p^\gamma(U_\gamma)) = \theta(\gamma U_\gamma)$, so that we have: $\beta^k \theta(U_\gamma^*) = \gamma^{-k} \theta(U_\gamma^*)$,

$$\beta^k(v_\gamma) = \gamma^{-k} e_{p1}^1 \theta(U_\gamma^*) + \sum_{j=1}^{p-1} e_{j,j+1}^1,$$

$$v_\gamma(\beta^p v_\gamma) \dots \beta^{-1}(v_\gamma) = \sum \gamma^j e_{jj}^1 \theta(U_\gamma^*) = U_\gamma \theta(U_*^\gamma).$$

(c) We just have to prove that $(s_p^\gamma)^q$ is outer for $q \in \{1, \dots, p-1\}$. To do this,

[3]) In all sums like $\sum_1^p e_{j,j+1}$ one takes $e_{p,p+1} = e_{p,1}$, say more generally that $e_{i+p_1, j+p_2} = e_{i,j}$ whenever p_1 and p_2 are multiples of p.

note that $v_{\gamma''}$ commutes with $\theta^j(U_{\gamma'})$ for $j \geqq 1$, $\{\gamma'\}^p = 1$, $(\gamma'')^p = 1$. Also $v_{\gamma''} U_{\gamma'} v_{\gamma''}^* = \gamma' U_{\gamma'}$ as seen above, so that:

$$s_p^\gamma(\theta^n(U_{\gamma'})) = \gamma' \theta^n(U_{\gamma'}), \quad \forall n \in \mathbf{N}, \quad \forall \gamma', \ \gamma'^p = 1.$$

This shows that for $q \in \{1, \ldots, p-1\}$ we have $\|(s_p^\gamma)^q \theta^n(U_{\gamma'}) - \theta^n(U_{\gamma'})\|_2 = |\gamma'^q - 1|$, hence that $(s_p^\gamma)^q$ cannot be an inner automorphism because the sequence $(\theta^n(U_{\gamma'}))_{n \in \mathbf{N}}$ is a central sequence in R.

We can now state an important consequence of theorem 1.5 and proposition 1.6:

Theorem 1.11. *Let R be the hyperfinite factor of type* II_1. *Let $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$ with $\gamma^p = 1$ and ε be a probability measure on* $\mathbf{T}$ *such that* Support $\varepsilon \subset \{n$ th *roots of $\lambda_0\}$ for some $\lambda_0 \in \mathbf{T}$ and $n \in \mathbf{N}$. Then there exists some periodic automorphism $\alpha \in$* Aut R, *satisfying the conditions $p_0(\alpha) = p$, $\gamma(\alpha) = \gamma$, $\varepsilon(\alpha) = \varepsilon$. Moreover let β be an inner automorphism of R such that $\varepsilon(\beta^{p\,\mathrm{Order}\,\gamma}) = \varepsilon$ then any $\alpha \in$* Aut R *periodic, with invariants $p_0(\alpha) = p$, $\gamma(\alpha) = \gamma$, $\varepsilon(\alpha) = \varepsilon$ is conjugate to*

$$s_p^\gamma \otimes \beta \in \mathrm{Aut}\, R \otimes R.$$

Proof. We just have to check that the outer invariant of $s_p^\gamma \otimes \beta$ is (p, γ) which is easy and to check that its inner invariant is ε. But $(s_p^\gamma \otimes \beta)^{p\,\mathrm{Order}\,\gamma} = 1 \otimes \beta^{p\,\mathrm{Order}\,\gamma}$.

Q.E.D.

II. Minimal periodic automorphisms

Throughout N is a factor, countably decomposable for simplicity. For $\alpha \in$ Aut N let Sp α be the spectrum of α in the Banach algebra $B(N)$ of weakly continuous linear mappings from N to N. Then Sp α is a closed subset of $\mathbf{T} = \{z \in \mathbf{C}, |z| = 1\}$ and is equal to the spectrum in the sense of [1] [4] of the representation $n \to \alpha^n$ of $\mathbf{Z}$ on N [4]) (cf. [4] 2.3.8).

For any nonzero projection $e \in N^\alpha$ we put as in [4] p.170, $\alpha^e = \alpha$ restricted to N_e, and we have by [4] 2.2.1 and 2.3.17 that

(2.1) $$\Gamma(\alpha) = \bigcap_{e \in N^\alpha} \mathrm{Sp}\, \alpha^e = \bigcap_{W \text{ unitary in } N} \mathrm{Sp}\, {}_W\alpha$$

where ${}_W\alpha = \mathrm{Ad}\, W \cdot \alpha$ by definition.

By [4] thm. 2.2.4, $\Gamma(\alpha)$ is a closed subgroup of $\mathbf{T}$ and by [4] thm. 2.3.1 we have:

(2.2) $$\Gamma(\alpha)^\perp = \{n \in \mathbf{Z}, \alpha^n \text{ is inner: } \alpha^n(x) = uxu^*, \ \forall x \in N \text{ with } u \in N^\alpha\}.$$

[4]) Identifying $\mathbf{T}$ with the dual group of $\mathbf{Z}$, by $(n, \lambda) = \lambda^n$, $\lambda \in \mathbf{T}$, $n \in \mathbf{Z}$.

Proposition 2.3. *Let* α *be a periodic automorphism of* N *and let* $p_m(\alpha) = p_0(\alpha) \cdot \text{Order}\,\gamma(\alpha)$, *where* (p_0, γ) *are the outer invariants of* α. *Then* $p_m(\alpha)$ *is the smallest period of automorphisms outer conjugate to* α. *(We call* $p_m(\alpha)$ *the minimal period of* α.*)*

Proof. As $\alpha^{p_m(\alpha)} = \operatorname{Ad} U^{\text{Order}\,\gamma(\alpha)}$, where $\alpha^{p_0(\alpha)} = \operatorname{Ad} U$, we see that $p_m(\alpha) \in \Gamma(\alpha)^\perp$. Conversely, if $q \in \Gamma(\alpha)^\perp$ then q is a multiple $np_0(\alpha)$ of $p_0(\alpha)$ and necessarily $\gamma(\alpha)^n = 1$ so that q is a multiple of $p_m(\alpha)$. Using [4] corollary 2.3.11 we get proposition 2.3 because

$$(2.4) \qquad \Gamma(\alpha)^\perp = \{p_m(\alpha)\mathbf{Z}\}, \quad \Gamma(\alpha) = \{\lambda \in \mathbf{C} \colon \lambda^{p_m(\alpha)} = 1\}.$$

The following equivalent conditions define the minimal periodic automorphisms:

Theorem 2.5. *Let* α *be a periodic automorphism of* N, *then the following conditions are equivalent:*

(a) Period of α = Minimal period of α,

(b) $\operatorname{Sp} \alpha = \Gamma(\alpha)$,

(c) N^α *is a factor,*

(d) *For* $n \in \{1, \ldots, (\text{period}\ \alpha) - 1\}$ *and* $\alpha^n = \operatorname{Ad} U$, $U \in N$ *one has* $\alpha(U) \neq U$.

Proof. That (a)$\Leftrightarrow$(b) follows from 2.4; that (b)$\Leftrightarrow$(c) is a corollary of [4] thm. 2.4.1, also (a)$\Leftrightarrow$(d) follows from 2.2.

Corollary 2.6. *Let* α *be a minimal periodic automorphism of* N *with minimal period* p, *then*

(a) *A unitary* $U \in N$ *is of the form* $v^* \alpha(v)$, v *unitary in* N, *if and only if* $U\alpha(U) \cdots \alpha^{p-1}(U) = 1$.

(b) *Any minimal periodic* $\beta \in \operatorname{Aut} N$ *which is outer conjugate to* α *is conjugate to* α.

Proof. The condition (a) is clearly necessary since for any v one has

$$v^* \alpha(v) \alpha\big(v^* \alpha(v)\big) \cdots \alpha^{p-1}\big(v^* \alpha(v)\big) = v^* v = 1.$$

To prove that it is sufficient, let ([4] lemma 2.2.6) β be the automorphism of $N \otimes F_2$ [5]) such that:

$$\beta(x \otimes e_{11}) = \alpha(x) \otimes e_{11}, \quad \beta(x \otimes e_{22}) = U\alpha(x)U^*, \quad \beta(1 \otimes e_{21}) = U \otimes e_{21}.$$

Then the condition (a) and the computation in [4] p. 176 show that $\beta^p = 1$. Hence, as $\Gamma(\beta) = \Gamma(\alpha) = \{p\mathbf{Z}\}$ we see that β is minimal periodic.

[5]) F_2 is a type I_2 factor with a system of matrix units $(e_{ij})_{i,j=1,2}$.

If N is finite, then $1\otimes e_{11}$ and $1\otimes e_{22}$ have the same trace in the finite factor $(N\otimes F_2)^\beta$ and hence are equivalent in $(N\otimes F_2)^\beta$.

If N is properly infinite, then so are N^α and $N^{(U^\alpha)}$ which means that $1\otimes e_{11}$ and $1\otimes e_{22}$ are properly infinite, and hence equivalent, in $(N\otimes F_2)^\beta$.

In all cases there hence exists a partial isometry $v^*\otimes e_{21}\in(N\otimes F_2)^\beta$ with $1\otimes e_{11}$ as initial support and $1\otimes e_{22}$ as final support. But then $\beta(v^*\otimes e_{21})=v^*\otimes e_{21}$ means $U\alpha(v^*)=v^*$ so that v is the required unitary. (b) We can assume that $\beta={}_W\alpha$ for some W. As $\Gamma(\beta)=\Gamma(\alpha)$ we see that the period of β is equal to $p=$period α. It follows that $W\alpha(W)\dots\alpha^{p-1}(W)$ is a scalar. Replacing W by λW, $\lambda\in\mathbf{C}$, for a suitable λ we can assume that $W\alpha(W)\dots\alpha^{p-1}(W)=1$. Then statement (a) of the corollary shows that β is conjugate to α.

Corollary 2.7. *Let α be a minimal periodic automorphism of N.*

(a) *Let e_1, e_2 be two projections in N^α which are equivalent relative to N. Then for any $\lambda\in\mathrm{Sp}\,\alpha$ there exists a partial isometry $U\in N$ such that:*

$$\alpha(U)=\lambda U,\ U^*U=e_1,\ UU^*=e_2.$$

(b) *If N is continuous, then for each integer m dividing* period $\alpha=p$ *and each λ, $\lambda^m=1$ there exists a system of $m\times m$ matrix units $e_{ij}\in N$ such that*

$$\alpha(e_{ij})=\lambda^{i-j}e_{ij}.\text{[6]}$$

Proof. (a) As we have seen above, either N is finite and $e_1\sim e_2(N^\alpha)$ or N is properly infinite and still $e_1\sim e_2(N^\alpha)$ so there exists a partial isometry $v\in N^\alpha$, $v^*v=e_1$, $vv^*=e_2$. Now α^{e_1} has period at most the period of α while $\Gamma(\alpha^{e_1})=\Gamma(\alpha)$ by [4] 2.3.3. Hence α^{e_1} is minimal periodic with the same period as α. But then by corollary 2.6 (a), there exists a unitary operator $W\in N^{e_1}$ such that $\alpha(W)=\lambda W$ (apply 2.6 (a) to $U=\lambda$). We then have $W^*W=e_1$, $WW^*=e_1$ and $\alpha(W)=\lambda W$ so that $U=vW$ satisfies the condition (a) of 2.7. (b) The factor N^α is continuous because a minimal projection in N^α is automatically minimal in N.

So we let $(e_j)_{j=1,\dots,m}$ be a family of m projections of N^α equivalent in N. Then by 2.7 (a), let U_j satisfy $U_j\in N$, $U_j^*U_j=e_j$, $U_jU_j^*=e_{j+1}$ and $\alpha(U_j)=\lambda U_j$, for $j=1,2,\dots,m-1$.

It follows that $e_{j+1,j}=U_j$ generates a system of matrix units satisfying the required conditions. Q.E.D.

Corollary 2.8. *Let α and β be periodic automorphisms of a factor N of type II_1 with canonical trace τ. Then α and β are conjugate if and only if they are outer conjugate and the inner automorphisms $\alpha^{p_m(\alpha)}$ and $\beta^{p_m(\alpha)}$ are conjugate.*

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[6]) Throughout $i\pm j$ for $i,j\in\{1,\dots,m\}$ means $i\pm j$ modulo m.

In other words two elements of an outer conjugacy class are conjugate if and only if they have the same inner invariant.

Proof. The condition is clearly necessary. To prove that it is sufficient first note that if α and β are outer conjugate we have $p_m(\alpha)=p_m(\beta)=p$ for some $p\in\mathbf{N}$. Write now $\alpha^p=\mathrm{Ad}\,U$, $U=\sum_{i=1}^{k}\lambda_i e_i$ where the e_i are projections belonging to the center of N^α (we use 2.4 and 2.2), with say $\tau(e_i)=\mu_i$, and the λ_i are complex numbers of modulus 1. For each λ_i choose a pth root and call it $\lambda_i^{1/p}$, then $U^{1/p}=$ $=\sum_{i=1}^{k}\lambda_i^{1/p}e_i$ belongs to N^α so that $\tilde\alpha=\mathrm{Ad}\,U^{-1/p}\alpha$ satisfies $\tilde\alpha^p=\mathrm{Ad}\,U^*\alpha^p=1$.

Write then $\beta^p=\mathrm{Ad}\,v$, with $v=\sum_{i=1}^{k}\lambda_i f_i$ where the λ_i are the same as the λ_i used above for U and where for each i, f_i is a projection (belonging to N^β) which is equivalent to e_i relative to the factor N. We have used the fact that α^p and β^p are conjugate inner automorphisms of N.

Choose $v^{1/p}=\sum_{i=1}^{k}\lambda_i^{1/p}f_i$ and put as above:

$$\tilde\beta=\mathrm{Ad}\,v^{-1/p}\beta.$$

We have $\tilde\beta^p=1$. Hence $\tilde\alpha$ and $\tilde\beta$ are outer conjugate and minimal periodic, so that by 2.6 they are conjugate, say $\tilde\beta=\sigma\tilde\alpha\sigma^{-1}$, $\sigma\in\mathrm{Aut}\,N$. Now $\alpha=\tilde\alpha\cdot\mathrm{Ad}\,U^{1/p}=\mathrm{Ad}\,U^{1/p}\tilde\alpha$, and:

$$\sigma\alpha\sigma^{-1}=\tilde\beta\,\mathrm{Ad}\,\sigma(U^{1/p})=\mathrm{Ad}\,\sigma(U^{1/p})\tilde\beta.$$

As we have $\beta=\mathrm{Ad}\,v^{1/p}\tilde\beta$ we just have to find an automorphism θ of N commuting with $\tilde\beta$ and such that $\theta\sigma(U^{1/p})=v^{1/p}$. Both $\sigma(U^{1/p})$ and $v^{1/p}$ belong to $N^{\tilde\beta}$ and we look for θ as an inner automorphism defined by a unitary $X\in N^{\tilde\beta}$.

We have $\sigma(U^{1/p})=\sum_{i=1}^{k}\lambda_i^{1/p}\sigma(e_i)$, $v^{1/p}=\sum_{i=1}^{k}\lambda_i^{1/p}f_i$ so it is enough to check that for each i, $\sigma(e_i)$ is equivalent to f_i relative to $N^{\tilde\beta}$. But this is true because $\tau(e_i)=\tau(f_i)$ hence $\tau(\sigma(e_i))=\tau(f_i)$ for $i=1,\dots,k$.

III. Action of automorphisms of R on central sequences

Definition 3.1. [7] *Let N be a II_1 factor with canonical trace τ, and ω be a free ultrafilter on* $\mathbf{N}$.

(a) *We let $N_{\tau,\omega}$ be the quotient of $l^\infty(\mathbf{N},N)$ by the two sided ideal $J_\omega=\{(x_n)_{n\in\mathbf{N}}$, $x_n\to 0$* strongly when *$n\to\omega\}$.*

(b) *We let N_ω be the commutant in $N_{\tau,\omega}$ of the image $\tilde N$ of N in $N_{\tau,\omega}$, where for $x\in N$, $\tilde x\in N_{\tau,\omega}$ is represented by the sequence $(x_n)_{n\in\mathbf{N}}$, $x_n=x$, $\forall n\in\mathbf{N}$.*

This definition is exactly the one given by McDuff in [7], except for a change of notations which matches with [5] part II. By [7] we know that $N_{\tau,\omega}$ is a II_1 factor with canonical trace τ_ω: $\tau_\omega((x_n)_{n\in\mathrm{N}})=\lim_{n\to\omega}\tau(x_n)$.

Let $\theta\in\mathrm{Aut}\,N$ then the automorphism $(x_n)_{n\in\mathrm{N}}\to(\theta(x_n))_{n\in\mathrm{N}}$ of $l^\infty(\mathrm{N}, N)$ leaves J_ω globally invariant and thus defines an automorphism $\theta_{\tau,\omega}$ of $N_{\tau,\omega}$. Moreover $\theta_{\tau,\omega}(\tilde{N})=\tilde{N}$ so that $\theta_{\tau,\omega}$ leaves N_ω globally invariant and thus defines an automorphism θ_ω of N_ω.

Now take $N=R$, the hyperfinite II_1 factor. We know that all hypercentral sequences on R are trivial [7], hence by [7] thm. 4, R_ω is a factor of type II_1.

Theorem 3.2. *Let α be an automorphism of the II_1 hyperfinite factor R, and ω be a free ultrafilter, then:*

(1) *α_ω is inner on R_ω if and only if α is inner on R, in which case $\alpha_\omega=1$.*

(2) *There exists a unitary $U\in R_{\tau,\omega}$ such that $(\alpha(x))^\sim=U\tilde{x}U^*$, $\forall x\in R$.*

Before we proceed and prove this theorem, let us note one consequence for periodic automorphisms:

Corollary 3.3. *Let $\alpha\in\mathrm{Aut}\,R$ be periodic, with* Outer period $\alpha=$period $\alpha=n$, *then there exists a unitary $v\in R_{\tau,\omega}$ such that:*

$$\alpha_{\tau,\omega}(v)=v;\quad v^n=1;\quad (\alpha(x))^\sim=v\tilde{x}v^*,\quad \forall x\in R.$$

Proof. Let U be a unitary, $U\in R_{\tau,\omega}$ such that $(\alpha(x))^\sim=U\tilde{x}U^*$, $x\in R$. We hence have that $\alpha_{\tau,\omega}(\tilde{x})=U\tilde{x}U^*$, $\forall x\in R$ and replacing x by $\alpha^{-1}(x)$: $\alpha_{\tau,\omega}(\tilde{x})=$ $=\alpha_{\tau,\omega}(U)\tilde{x}\alpha_{\tau,\omega}(U^*)$, $\forall x\in R$. Then $U^*\alpha_{\tau,\omega}(U)\in(\tilde{R})'\cap R_{\tau,\omega}=R_\omega$.

Put $w=U^*\alpha_{\tau,\omega}(U)$. We have $w\alpha_{\tau,\omega}(w)\dots\alpha_{\tau,\omega}^{n-2}(w)\alpha_{\tau,\omega}^{n-1}(w)=1$. Now as Outer period $\alpha=n$ we have Outer period $\alpha_\omega=n$ using part (1) of theorem 3.2. So it follows from corollary 2.6 that this $w\in R_\omega$ such that $w\alpha_\omega(w)\dots\alpha_\omega^{n-1}(w)=1$ can be written $w=X^*\alpha_\omega(X)$ for some unitary $X\in R_\omega$.

Put $Y=UX^*$. Then $\alpha_{\tau,\omega}(Y)=\alpha_{\tau,\omega}(U)\alpha_{\tau,\omega}(X^*)=Uww^*X^*=Y$ and $(\alpha(x))^\sim=$ $=Y\tilde{x}Y^*$, $\forall x\in R$, because $X^*\in(\tilde{R})'$.

Now $\alpha_{\tau,\omega}^n(\tilde{x})=Y^n\tilde{x}(Y^*)^n$, $\forall x\in R$, so that $Y^n\in R_\omega$. As $R_\omega^{\alpha_\omega}$ is a von Neumann algebra, there exists a $Z\in R_\omega^{\alpha_\omega}$ such that $Z^n=Y^n$ and Z is in the von Neumann algebra generated by Y^n in $R_\omega^{\alpha_\omega}$. In particular Z and Y commute as elements of $R_{\tau,\omega}$. Put $U'=YZ^*$ then we have:

$$\alpha_{\tau,\omega}(U')=\alpha_{\tau,\omega}(Y)\alpha_\omega(Z^*)=YZ^*=U',$$
$$U'^n=Y^n(Z)^{-n}=1,\quad U'\tilde{x}U'^*=Y\tilde{x}Y^*=(\alpha(x))^\sim,\quad \forall x\in R.\qquad \text{Q.E.D.}$$

The proof of the theorem, part (1), relies on the following simple adaptation of the proof given in [10] p. 156—157 of the derivation theorem which can also be found in [3] with $\|\ \ \|$ instead of $\|\ \ \|_2$.

Lemma 3.4. *Let P be a factor of type* II_1, *and K be a finite dimensional subfactor of P. Let $\alpha \in \mathrm{Aut}\, P$, then if* $\operatorname{Sup}_{U \text{ unitary in } K' \cap P} \|\alpha(U) - U\|_2 < 1$ *the automorphism α is inner.*

Proof. Let U_0 be the unitary group of $K' \cap P$. Then, exactly as in [10] p. 156—157 we define an action of U_0 on the vector space P by the formula

$$\varphi_u(x) = ux\alpha(u^*), \quad x \in P, \quad u \in U_0.$$

As $\|\varphi_u(x)\|_2 = \|x\|_2$, $\forall x \in P$ we can extend this action to an action of U_0 on $L^2(P, \tau)$ where τ is the canonical trace of P. If the hypothesis of the lemma is satisfied the orthogonal projection y of 0 on $\overline{\mathrm{Conv}}\{\varphi_u(1), u \in U_0\}$ [7]) is different from 0 and is a fixed point for φ_u, for all $u \in U_0$. In other words we have $uy = y\alpha(u)$, $\forall u \in U_0$, hence $xy = y\alpha(x)$ $\forall x \in K' \cap P$. Now there exists a unitary $v \in P$ such that ${}_v\alpha = \mathrm{Ad}\, v \cdot \alpha$ is of the form $1_K \otimes \beta$ where $\beta \in \mathrm{Aut}\,(K' \cap P)$, and we have:

$$xyv^* = yv_v^* \alpha(x), \quad \forall x \in K' \cap P.$$

Let $yv^* = \sum e_{ij} \otimes y_{ij}$, with $e_{ij} \in K$ and $y_{ij} \in K' \cap P$, then if the e_{ij} are matrix units in K we get $xy_{ij} = y_{ij}\beta(x)$, $\forall x \in K' \cap P$, $\forall i, j$. It follows then that there exists a nonzero $z \in K' \cap P$ such that

$$xz = z\beta(x), \quad \forall x \in K' \cap P.$$

Hence by [9] $\beta = ({}_v\alpha$ restricted to $K' \cap P)$ is an inner automorphism, so that ${}_v\alpha$ is inner on P, being identity on K, and finally α is inner on P.

Proof of part (1) of theorem 3.2. If α is an inner automorphism then easily $\alpha_\omega = 1$. Let α be an outer automorphism, and u_n be a sequence of unitaries of R. We construct a central sequence $(v_n)_{n \in \mathbf{N}}$ of unitaries such that $\|u_n v_n u_n^* - v_n\|_2 \to 0$ as $n \to \infty$ and $\|\alpha(v_n) - v_n\|_2 \geqq \frac{1}{2}$, $\forall n$. It follows that for any unitary $u \in R_{\tau, \omega}$ there exists a unitary $v \in R_\omega$ such that $uvu^* = v$ while $\alpha(v) \neq v$. This will hence show that α_ω is not inner on R_ω.

To construct the sequence $(v_n)_{n \in \mathbf{N}}$, let K_n be an increasing sequence of finite dimensional subfactors of R such that: $u_n \overset{1/n}{\in} K_n$, [8]) $\forall n \in \mathbf{N}$, $(\bigcup_{n \in \mathbf{N}} K_n)^- = R$. Let then, for each n, v_n be a unitary in K_n' such that $\|\alpha(v_n) - v_n\|_2 \geqq \frac{1}{2}$ (we apply lemma 3.4).

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[7]) The canonical image of the C^* unit ball of P in $L^2(P, \tau)$ is weakly closed and contains the $\varphi_u(1)$, $u \in U_0$. So y belongs to this unit ball.

[8]) $x \overset{\varepsilon}{\in} K$ means, as in [7], the existence of some $y \in K$ such that $\|x - y\|_2 < \varepsilon$.

Clearly $(v_n)_{n\in\mathbf{N}}$ is a central sequence in R and $\|[u_n, v_n]\|_2 \leqq \frac{2}{n}$, $\forall n\in\mathbf{N}$, so that $\|u_n v_n u_n^* - v_n\|_2 \xrightarrow[n\to\infty]{} 0$.

Q.E.D.

Proof of part (2) of theorem 3.2. Let $(K_n)_{n\in\mathbf{N}}$ be an increasing sequence of finite dimensional subfactors of R. For each $n\in\mathbf{N}$ there exists a unitary u_n' such that $\alpha(K_n)=u_n' K_n u_n'^*$ hence a unitary u_n such that $\alpha(x)=u_n x u_n^*$, $\forall x\in K_n$.

It follows that $\alpha(x)=\lim_{n\to\infty} u_n x u_n^*$, $\forall x\in \bigcup_{n=1}^{\infty} K_n$ and hence for all $x\in R$, provided $\bigcup_{n=1}^{\infty} K_n$ is strongly dense in R.

Q.E.D.

IV. Some technical lemmas

Lemma 4.1. *Let* $\varepsilon\in]0,1[$, *and* N *be a factor of type* II_1. *Let* $e_1, \dots, e_n$ *be projections of* N *such that* $\left\|\sum_{j=1}^{n} e_j - 1\right\|_2 \leqq \varepsilon$. *Then*

(a) $f_j = \bigvee_1^j e_l - \bigvee_1^{j-1} e_k$ *is a family of pairwise orthogonal projections such that* $\|f_j - e_j\|_2 \leqq 10 n\varepsilon^{1/4}$.

(b) *There exists a family of pairwise orthogonal projections* $E_j \sim e_j$ *with* $\|E_j - e_j\|_2 \leqq 14 n\varepsilon^{1/8}$, *provided* $\Sigma\tau(e_j)\leqq 1$.

Proof. (a) Put $T_j=\sum_1^j e_l$, and $F_j=\bigvee_1^j e_l$. We have $F_j=$Support T_j. Let f^j be the spectral projection of T_j corresponding to the interval $[1+\sqrt{\varepsilon}, \infty[$. As we have $T_j\leqq T_n$, we get $\tau(f^j)\leqq\tau(f^n)$ by the minimax theorem, hence $\tau(f^j)\leqq\varepsilon$. Also f^j commutes with T_j and F_j and as $\|(1+\sqrt{\varepsilon})F_j - T_j\|\leqq n$, we have $\tau((1-f^j)((1+\sqrt{\varepsilon})F_j - T_j)) \leqq \tau((1+\sqrt{\varepsilon})F_j - T_j) + n\tau(f^j) \leqq \sqrt{\varepsilon}+n\varepsilon$. (Because $\tau(F_j)\leqq \sum_1^j \tau(e_{j'})=\tau(T_j)$.) As $0\leqq(1-f^j)((1+\sqrt{\varepsilon})F_j-T_j)\leqq 1+\sqrt{\varepsilon}\leqq 2$ we have $\|(1-f^j)((1+\sqrt{\varepsilon})F_j-T_j)\|_2 \leqq \sqrt{2}(n+1)^{1/2}\varepsilon^{1/4}$.

But $\|f^j((1+\sqrt{\varepsilon})F_j-T_j\|_2\leqq n(\tau(f^j))^{1/2}\leqq n\varepsilon^{1/4}$, so that $\|F_j-T_j\|_2\leqq\|(1+\sqrt{\varepsilon})F_j-T_j\|_2+\sqrt{\varepsilon}\leqq(n\sqrt{2}+(n+1)^{1/2}\sqrt{2}+1)\varepsilon^{1/4}$. As $T_j-T_{j-1}=e_j$ and $F_j-F_{j-1}=f_j$ we get (a).

(b) We have $\Sigma\tau(e_j)\leqq 1$. Take f_j as in (a). Let $I=\{j\in\{1,\dots,n\}, \tau(f_j)\leqq\tau(e_j)\}$ and $J=\{j\in\{1,\dots,n\}, \tau(f_j)>\tau(e_j)\}$. Let for each $j\in J$, f_j' be a subprojection of f_j such that $\tau(f_j')=\tau(f_j)-\tau(e_j)$. Then the f_j' are pairwise orthogonal with $\sum_{j\in J}\tau(f_j')+\tau(1-\bigvee_{j=1}^{n} f_j) = \sum_{j\in J}\tau(f_j')+1-\sum_{j=1}^{n}\tau(f_j) = 1-\sum_{j\in I}\tau(f_j)-\sum_{j\in J}\tau(e_j) \geqq \sum_{j\in I}(\tau(e_j)-\tau(f_j))$. Let $(f_j'')_{j\in I}$ be a family of pairwise orthogonal projections, with $f_j''\leqq\sum_{j\in J} f_j' + 1-\sum_{j=1}^{n} f_j$, $\tau(f_j'')=\tau(e_j-f_j)$, $\forall j\in I$. Put $E_j=f_j+f_j''$ for $j\in I$, $R_j=f_j-f_j'$ for

$j\in J$. Then each E_j is a projection equivalent to e_j, the E_j's are pairwise orthogonal, and

$$\|E_j-f_j\|_2 \leqq |\tau(e_j)-\tau(f_j)|^{1/2} \leqq 4n^{1/2}\varepsilon^{1/8}, \quad \|E_j-e_j\|_2 \leqq (10n+4n^{1/2})\varepsilon^{1/8} \leqq 14n\varepsilon^{1/8}.$$

Lemma 4.2. *Let $n, m\in\mathbf{N}$ with n dividing m. Let α be a minimal periodic automorphism of period m, of a II_1 factor N. Let $\eta\in]0, 1/m[$ and $\lambda\in\mathbf{C}$, $\lambda^n=1$. Let $(u_j)_{j=1,\dots,n-1}$ be a family of $n-1$ elements of N of norm less than 1 such that*

(a) $\|\alpha(u_j)-\lambda u_j\|_2\leqq\eta$, $j\in\{1, \dots, n-1\}$,

(b) $\left\|\left(\sum_1^{n-1} u_j^* u_j\right)+u_{n-1}u_{n-1}^*-1\right\|_2\leqq \eta$,

(c) $\|u_j^* u_j-(u_j^* u_j)^2\|_2 \leqq \eta$ *for* $j\in\{1, \dots, n-1\}$, *and*

$\|u_{n-1}u_{n-1}^*-(u_{n-1}u_{n-1}^*)^2\|_2 \leqq \eta$,

(d) $\|u_j u_j^*-u_{j+1}^* u_{j+1}\|_2 \leqq \eta$ *for* $j = 1, \dots, n-2$.

Then there exists a system of matrix units $(e_{ij})_{i,j\in\{1,\dots,n\}}$ of N such that $\alpha(e_{ij})=$ $=\lambda^{i-j}e_{ij}$ and

$$\|u_j-e_{j+1,j}\|_2\leqq 142n(m\eta)^{1/256} \quad \textit{for all} \quad j\in\{1, \dots, n-1\}.$$

Proof. For $x\in N$, we put $x^\lambda=\frac{1}{m}\sum_{j=1}^{m}\bar\lambda^j\alpha^j(x)$. Then if $\|x\|\leqq 1$, $\|\alpha(x)-\lambda x\|_2\leqq\eta$ one has $\|x^\lambda\|\leqq 1$, $\alpha(x^\lambda)=\lambda x^\lambda$ and $\|x-x^\lambda\|_2\leqq\frac{1}{m}\sum_1^m k\eta\leqq\frac{m-1}{2}\eta$. Take $v_j=u_j^\lambda$, then we have

(e) $\|u_j-v_j\|_2 \leqq \frac{m-1}{2}\eta$, $\|u_j^* u_j-v_j^* v_j\| \leqq (m-1)\eta$,

$\|u_{n-1}u_{n-1}^*-v_{n-1}v_{n-1}^*\| \leqq (m-1)\eta$ and the v_j satisfy a condition like (b) with $nm\eta$ instead of η, like (c) with $3m\eta$ and (d) with $2m\eta$.

Put $T_j=v_j^* v_j$ for $j=1, \dots, n-1$ and $T_n=v_{n-1}v_{n-1}^*$. Then we have $T_l\in N^\alpha$ $(l=1, \dots, n)$ and $\|T_l^2-T_l\|_2\leqq 3m\eta$ $(l=1, \dots, m)$.

Then by [6] p. 273—274 there exists for $j\in\{1, \dots, n\}$ a spectral projection F_j of T_j, $F_j\in N^\alpha$ such that:

(f) $\|T_j-F_j\|_2\leqq 8(m\eta)^{1/2}$, $\|T_j^{1/2}-F_j\|_2\leqq 6(m\eta)^{1/4}$.

If for $j\in\{1, \dots, n-1\}$ we let $v_j=V_j T_j^{1/2}$ be the polar decomposition of v_j we get:

(f′) $\|v_j-V_j F_j\|_2\leqq 6(m\eta)^{1/4}$.

Put $a=\mathrm{Inf}(1/n, \tau(F_1), \dots, \tau(F_n))$. We have $|\tau(F_j)-\tau(T_j)| \leqq 8(m\eta)^{1/2}$, $j\in\{1, \dots, n\}$;

hence using condition (b) for the v's:

$$\left|\sum_{j=1}^{n} \tau(F_j)-1\right| \leqq \left|\sum_{j=1}^{n} \tau(T_j)-1\right|+8n(m\eta)^{1/2} \leqq nm\eta+8n(m\eta)^{1/2}\leqq 9n(m\eta)^{1/2}.$$

(Because $m\eta\leqq 1$.)

For $j<n-1$

$$|\tau(F_j)-\tau(F_{j+1})|\leqq|\tau(v_j v_j^*)-\tau(v_{j+1}^* v_{j+1})|+16(m\eta)^{1/2}\leqq 2m\eta+16(m\eta)^{1/2}\leqq 18(m\eta)^{1/2}.$$

Moreover $|\tau(F_{n-1})-\tau(F_n)|\leqq 18(m\eta)^{1/2}$. So that

$$\left|\tau(F_j)-\frac{1}{n}\right| \leqq \left|\tau(F_j)-\frac{1}{n}\sum_{i=1}^{n}\tau(F_i)\right|+9(m\eta)^{1/2}\leqq \frac{n-1}{2}18(m\eta)^{1/2}+9(m\eta)^{1/2}\leqq 9n(m\eta)^{1/2}.$$

Hence $\left|a-\frac{1}{n}\right|\leqq 9n(m\eta)^{1/2}$ and $|\tau(F_j)-a|\leqq 18n(m\eta)^{1/2}$. For each j, let E_j be a projection in N^α with $E_j\leqq F_j$, $\tau(E_j)=a$. Put $W_j=V_jE_j$, $j<n$. Then

$$\|W_j-v_j\|_2 \leqq \|F_j-E_j\|_2+\|v_j-V_jF_j\|_2 \leqq 5n^{1/2}(m\eta)^{1/4}+6(m\eta)^{1/4} \leqq (16n)(m\eta)^{1/4},$$
$$\|W_{n-1}W_{n-1}^*-T_n\|_2 \leqq 32n(m\eta)^{1/4};$$

for $j<n-1$,

$$\|W_jW_j^*-W_{j+1}^*W_{j+1}\|_2 \leqq 4\cdot 16n(m\eta)^{1/4}+2m\eta \leqq 66n(m\eta)^{1/4}.$$

Now we have $\tau(E_j)=a\leqq\frac{1}{n}$,

$$\left\|\sum_{j=1}^{n}E_j-1\right\|_2\leqq 5n^{3/2}(m\eta)^{1/4}+8n(m\eta)^{1/2}+\left\|\sum_{j=1}^{n}T_j-1\right\|_2\leqq 14n^{3/2}(m\eta)^{1/4}.$$

Hence there exists a system G_j of pairwise orthogonal projections of N^α, with $\tau(G_j)=\tau(E_j)=a$, and $\|G_j-E_j\|_2\leqq 22n^2(m\eta)^{1/32}$ (Lemma 4.1). For each $j<n$, we let, using [6] lemma 7 p. 275, X_j be a unitary in N^α such that $X_jG_jX_j^*=E_j$, Y_j be unitary in N^α with $Y_jW_jW_j^*Y_j^*=G_{j+1}$ and such that:

$$\|X_j-1\|_2 \leqq 50n^{1/4}(m\eta)^{1/256}, \quad \|Y_j-1\|_2 \leqq 72n^{1/4}(m\eta)^{1/256}.$$

Choose n pairwise orthogonal projections G_j' such that $G_j'\in N^\alpha$, $\tau(G_j')=\frac{1}{n}-a$, $G_j'\leqq 1-\sum_{k=1}^{n}G_k$, and $n-1$ partial isometries $(U_j')_{j=1,n-1}$ where $U_j'^*U_j'=G_j'$, $U_j'U_j'^*=G_{j+1}'$, $\alpha(U_j')=\lambda U_j'$ (apply 2.7a). Put $U_j=Y_jW_jX_j+U_j'$ for $j=1,\dots,n-1$. Then U_j is a partial isometry with initial support G_j+G_j' and final support $G_{j+1}+G_{j+1}'$.

Also $\alpha(U_j)=\lambda U_j$ and we have:

$$\|U_j-W_j\|_2 \leqq 122\left(n^{1/4}(m\eta)^{1/256}\right)+\left(\frac{1}{n}-a\right)^{1/2}, \quad \text{and using (e),}$$

$$\|U_j-u_j\|_2 \leqq \frac{m-1}{2}\eta+16n(m\eta)^{1/4}+122n^{1/4}(m\eta)^{1/256}+3n^{1/2}(m\eta)^{1/4} \leqq$$

$$\leqq (1+16n+122n^{1/4}+3n^{1/2})(m\eta)^{1/256} \leqq 142n(m\eta)^{1/256}. \qquad \text{Q.E.D.}$$

Lemma 4.3. *Let $n, m \in \mathrm{N}$ with n dividing m. Let α be a minimal periodic automorphism of period m of a II_1 factor N. Let $\delta>0$, and $(e_j)_{j=1,\dots,n}$ be a partition of unity in N such that $\alpha(e_j)=e_{j+1}$, $j=1,\dots,n$ $(e_{n+1}=e_1)$, and $U\in N$, $\|U\|\leqq 1$ with*

(1) $$\|U^{n-l}-(U^*)^l\|_2 \leqq \delta, \quad l=0,1,2,\dots,n-1 \quad (\text{read } (U^*)^0=1),$$

(2) $$\|\alpha(U)-U\|_2 \leqq \delta,$$

(3) $$\|Ue_iU^*-e_{i+1}\|_2 \leqq \delta \quad (i=1,\dots,n).$$

Then there exists a partition of unity $(E_j)_{j=1,\dots,n}$ of N, such that $\alpha(E_j)=E_{j+1}$ for $j=1,\dots,n$ $(E_{n+1}=E_1)$, and a unitary V, $V^n=1$, $V\in N^\alpha$ such that $VE_iV^*=E_{i+1}$ for $i=1,\dots,n$ and that

$$\|V-U\|_2 \leqq \varepsilon, \quad \|E_i-e_i\| \leqq \varepsilon, \quad i=1,\dots,n,$$

where $\varepsilon=143n^4(2mn^2\delta)^{1/256}$ (provided $2n^2\delta \leqq 1/m$).

Proof. Let $\lambda=\exp(i2\pi/n)$. Put $W=\sum\bar{\lambda}^j e_j$. Then W is a unitary of N such that $\alpha(W)=\sum\bar{\lambda}^j e_{j+1}=\lambda W$.

For $k\in\{0,\dots,n-1\}$, put $f_k=\frac{1}{n}\sum_{j=1}^{n-1}\bar{\lambda}^{jk}U^j$, where U^0 is taken to be 1. Then $U^l=\sum_{k=0}^{n-1}\lambda^{kl}f_k$ for $l=0,1,\dots,n-1$.

Moreover $\|f_k\|\leqq 1$ $(k=0,1,\dots,n-1)$ and

$$\|f_k^*-f_k\|_2 \leqq \frac{1}{n}\left\|\sum_{j=1}^{n-1}(\lambda^{-k})^{n-j}\left(U^{n-j}-(U^*)^j\right)\right\|_2+\frac{1}{n}\|U^n-1\|_2 \leqq 2\delta,$$

$$n^2f_k^*f_k=\sum_{j,l=0,\dots,n-1}(\lambda^k)^j(\lambda^k)^{-l}(U^*)^jU^l.$$

For $l\geqq j$ we have $\|(U^*)^jU^l-U^{l-j}\|_2 \leqq \|(U^*)^jU^l-U^{n-j}U^l\|_2+\|U^n-1\|_2 \leqq 2\delta$. And for $l<j$, we have $\|(U^*)^jU^l-U^{n-(j-l)}\|_2 \leqq \delta$. It follows that, in the above sum for $n^2f_k^*f_k$ one can replace each $(U^*)^jU^l$ by $U^{\underline{l-j}}$ where $\underline{l-j}=l-j$, mod n, and $0\leqq\underline{l-j}<n$ to get

$$\|n^2f_k^*f_k-n^2f_k\|_2 \leqq 2n^2\delta, \quad \|f_k^*f_k-f_k\|_2 \leqq 2\delta.$$

Put $U_j=Wf_j$ with f_j as above. Then as W is unitary we get: $U_j^*U_j=f_j^*f_j$, $\|U_j^*U_j-f_j\|_2\leqq 2\delta$. Also $\underline{U_{n-2}}\underline{U_{n-2}^*}=Wf_{n-2}f_{n-2}^*W^*$ so $\|U_{n-2}U_{n-2}^*-(U_{n-2}U_{n-2}^*)^2\|_2\leqq 4\delta$. We get $\|U_j^*U_j-(U_j^*U_j)^2\|_2\leqq 4\delta$ (because $\|U_j^*U_j-f_j^*\|_2\leqq 2\delta$, $\|U_j^*U_j-f_j\|\leqq 2\delta$).

Also $UWU^*=\Sigma\bar\lambda^jUe_jU^*$ so that $\|UWU^*-\lambda W\|_2\leqq n\delta$ and $\|UW^*U^*-\bar\lambda W^*\|_2\leqq n\delta$, $\|WUW^*-\bar\lambda U\|_2\leqq n\delta+\|U^*U-1\|_2\leqq (n+2)\delta$ because $\|U^*-U^{n-1}\|_2\leqq\delta$ and $\|U^n-1\|_2\leqq\delta$.

So for $j\in\{0,1,\dots,n-1\}$ we get $\|WU^jW^*-\bar\lambda^jU^j\|_2\leqq j(n+2)\delta$. Hence $\|Wf_kW^*-f_{k+1}\|_2\leqq\dfrac{1}{n}\Sigma\|WU^jW^*-\bar\lambda^jU^j\|_2\leqq\dfrac{(n-1)}{n}(n+2)\delta\leqq n^2\delta$. It follows that $\|U_kU_k^*-U_{k+1}^*U_{k+1}\|_2\leqq\|Wf_kW^*-f_{k+1}\|_2+4\delta\leqq(n^2+4)\delta$ for all $k=0,1,\dots,n-1$. Hence also $\|U_{n-2}U_{n-2}^*-f_{n-1}\|_2\leqq(n^2+2)\delta$ and $\left\|\left(\sum_0^{n-2}U_j^*U_j\right)+U_{n-2}U_{n-2}^*-1\right\|_2\leqq$ $\leqq 2(n-1)\delta+(n^2+2)\delta$ because $\sum_{j=0}^{n-1}f_j=1$. As $\alpha(W)=\lambda W$ and $\|\alpha(f_j)-f_j\|_2\leqq$ $\leqq\dfrac{1}{n}\sum_{l=0}^{n-1}\|U^l-\alpha(U^l)\|_2\leqq\dfrac{n-1}{2}\delta$ we have $\|\alpha(U_j)-\lambda U_j\|_2\leqq\dfrac{n-1}{2}\delta$.

We have shown that the family $(U_j)_{j=0,\dots,n-2}$ satisfies the conditions (a), (b), (c), (d) of Lemma 4.2 with $\eta=2n^2\delta$. By hypothesis $2n^2\delta\leqq\dfrac{1}{m}$ so that we can find a family $(e_{ij})_{i,j\in\{0,\dots,n-1\}}$ of partial isometries of N, such that $\alpha(e_{ij})=\lambda^{i-j}e_{ij}$ and

$$\|U_j-e_{j+1,j}\|_2\leqq 142n(2mn^2\delta)^{1/256}=\delta'\quad(j=0,1,\dots,n-2).$$

Moreover we have $U_{n-1}=Wf_{n-1}=(W^*)^{n-1}f_{n-1}$ and as $\|f_k^*W^*-W^*f_{k+1}^*\|_2$ is smaller than $n^2\delta$ for all k we get:

$$\|U_{n-1}-U_0^*U_1^*\dots U_{n-2}^*\|_2\leqq n(n^2\delta)+\sum_{j=1}^{n-1}\|(f_j^*)^2-f_j^*\|_2\leqq nn^2\delta+4n\delta,$$

$$\|U_{n-1}-e_{0,n-1}\|_2\leqq 2n^3\delta+n\delta'\leqq\delta''=143n^2(2mn^2\delta)^{1/256}.$$

Put $W_1=\sum_{j=0}^{n-1}e_{j+1,j}$. Then W_1 is a unitary such that $W_1^n=1$ and $\alpha(W_1)=\lambda W_1$. Put $E_j=\dfrac{1}{n}\sum_{l=0}^{n-1}\lambda^{jl}W_1^l$, for $j=0,\dots,n-1$, so that $W_1=\Sigma\bar\lambda^jE_j$ and the E_j are the spectral projections of W_1 corresponding to $\bar\lambda^j$, $j=0,\dots,n-1$. We have $\alpha(E_j)=E_{j+1}$ $(j=0,\dots,n-1)$.

Put $V=\Sigma\lambda^ke_{kk}$, then V is unitary, $V^n=1$, $\alpha(V)=V$ and $VE_jV^*=E_{j+1}$ $(j=0,1,\dots,n-1)$. (Because $VW_1V^*=\lambda W_1$.)

We have

$$\|W_1-W\|_2\leqq\left\|\sum_0^{n-1}(U_j-e_{j+1,j})\right\|_2\leqq n\delta'',$$

and hence

$$\|E_j - e_j\|_2 \leqq \frac{1}{n} \sum_0^{n-1} \|W_1^l - W^l\|_2 \leqq \left(\frac{n-1}{2}\right) n\delta''.$$

Also $\|V-U\|_2 \leqq \sum_{k=0}^{n-1} \|e_{kk} - f_k\|_2 \leqq n(2\delta') + (n-1)2\delta + n^2\delta \leqq 2\delta''$. Hence we get the conclusion of the lemma, taking $\varepsilon = n^2\delta'' = 143n^4(2mn^2\delta)^{1/256}$. Q.E.D.

V. Actions of finite cyclic groups, by outer automophisms, on the II_1 hyperfinite factor

The fact that for each n there exists only one action by outer automorphisms of $\mathbf{Z}/n$ on R follows from statement (b) of

Theorem 5.1. *Let R be the hyperfinite II_1 factor and $p, q \in \mathbf{N}$.*

(a) *Let $\alpha \in \operatorname{Aut} R$ be minimal periodic with* (outer period α)$=pq$ *then $\alpha \otimes s_q^1$ is conjugate to α, also $\alpha \otimes 1_R$ is conjugate to α.*

(b) *Any periodic $\alpha \in \operatorname{Aut} R$ such that* period $\alpha =$ outer period $\alpha = p$ *is conjugate to s_p^1.*

Proof. Let $(x_j)_{j\in\mathbf{N}}$ be a strongly dense sequence in the unit ball R_1 of R. We shall, in the proof of (a) and (b), construct a sequence (K_j) of type I_n subfactors of R pairwise commuting, with:

$$\|E_{K'_m}(x_l) - x_l\|_2 \leqq \frac{1}{2^m}, \quad l < m.\text{ [9]} \tag{5.2}$$

We recall that using [7], we then have for each l and

$$l' \geqq l, \quad x_l \overset{1/2^{l'-1}}{\in} \bigcap_{m \geqq l'} K'_m = \left((K_1 \cup \ldots \cup K_{l'})'' \cup \left(\bigcup_{j=1}^{\infty} K_j\right)'\right)''$$

because $(\prod_{m\geqq l'} E_{K'_m})(x)$ belongs to $\bigcap_{m\geqq l'} K'_m$.

Hence we know ([7]) that letting $K = \left(\bigcup_{k=1}^{\infty} K_j\right)''$ the factor R splits as the tensor product of K by its commutant K' in R.

(a) Let λ be an nth root of 1 where n is an integer dividing the outer period of α. We construct by induction on m a sequence K_m of pairwise commuting type I_n subfactors of R satisfying condition 5.2 and:

(5.3) $\alpha(K_m) = K_m$ and there exists a system of matrix units e_{ij}^m of K_m such that $\alpha(e_{ij}^m) = \lambda^{i-j} e_{ij}^m$.

The existence of K_1 follows from corollary 2.7.

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[9]) From now on, if K is a von Neumann subalgebra of R, we let E_K be the trace preserving conditional expectation of R onto K and let K' be the relative commutant of K in R.

Assume we have constructed the K_j's up to K_m included. We are looking for K_{m+1} such that:

(1) $K_{m+1} \subset (K_1 \cup \ldots \cup K_m)'$,

(2) K_{m+1} is generated by a system of $n \times n$ matrix units (e_{ij}) such that $\alpha(e_{ij}) = \lambda^{i-j} e_{ij}$ $(i, j = 1, \ldots, n)$.

(3) $\|[e_{i+1,i}, x_l]\|_2 < \varepsilon, \quad i = 1, \ldots, n-1; \quad l = 1, \ldots, m.$

Clearly, if ε is chosen small enough, the conditions (1), (2), (3) will imply conditions (5.2) and (5.3).

To get K_{m+1}, restrict to $P = (K_1 \cup \ldots \cup K_m)'$ which is a II_1 hyperfinite factor on which the restriction β of α has outer period equal to $p_0(\alpha)$. Then by corollary 2.7 and the fact that β_ω is minimal periodic of period $p_0(\alpha)$ (theorem 3.2.1) there exists a system $(v_j)_{j=1,\ldots,n}$ of partial isometries in P_ω such that, with $v_{n+1} = v_1$:

$$\beta_\omega(v_j) = \lambda v_j, \quad \sum_{j=1}^{n} v_j^* v_j = 1, \quad v_j v_j^* = v_{j+1}^* v_{j+1} \quad (j = 1, \ldots, n).$$

Then apply lemma 4.2 to construct the $(e_{j,k})_{j,k=1,\ldots,n}$ satisfying conditions (1), (2), (3) above.

Now let $K = \left(\bigcup_{j=1}^{\infty} K_j\right)''$. Take first $n = q$ and $\lambda = e^{i2\pi/q}$ then the restriction of α to K is obviously conjugate to s_q^1. Now $s_q^1 \otimes s_q^1$ is conjugate to s_q^1 so that $s_q^1 \otimes \alpha$ is conjugate to α, because $\alpha = \alpha/K \otimes \alpha/K'$.

Take then $n =$ outer period α and $\lambda = 1$, then the restriction of α to K is identity so that $\alpha \otimes 1_R$ is conjugate to α because $1_R \otimes 1_R = 1_R$.

(b) For each $n \in \mathrm{N}$ we choose a positive ε_n having the following property:

(5.4) Let U be a unitary in R with $U^p = 1$, $(e_j)_{j=1,\ldots,p}$ be a partition of unity in R with $U e_j U^* = e_{j+1}$, $j = 1, \ldots, p-1$, and K be the type I_p subfactor of R generated by U and the e_j's.

Then $x \in R$, $\|x\|_\infty \leqq 1$, $\|[x, U]\|_2 \leqq 2\varepsilon_n$, $\|[x, e_j]\|_2 \leqq 2\varepsilon_n$, $j = 1, \ldots, p$ implies

$$\|E_{K'}(x) - x\|_2 \leqq \frac{1}{2^{n+1}}.$$

We can moreover assume that $\varepsilon_{n+1} \leqq \varepsilon_n$ for each $n \in \mathrm{N}$ and $\varepsilon_n \xrightarrow[n\to\infty]{} 0$. Then we construct by induction a sequence $(K_n)_{n \in \mathrm{N}}$ of pairwise commuting type I_p subfactors of R satisfying condition 5.2 and

(5.5) For each $n \in \mathrm{N}$ one has $\alpha(K_n) = K_n$, α restricted to K_n equals $\mathrm{Ad}\, U_n$, U_n unitary in K_n and:

$$\left\|\alpha(x_j) - \left(\prod_1^n \mathrm{Ad}\, U_k\right)(x_j)\right\|_2 \leqq \varepsilon_n, \quad j = 1, \ldots, n.$$

We directly prove the existence of K_{n+1} assuming $K_1, \ldots, K_n$ have already been constructed. This will also show how to build K_1. Let $P=(K_1\cup\ldots\cup K_n)'\cap R$ and P_1 its unit ball. P is globally invariant under α, let $\beta=\alpha/P$, then $p_0(\beta)=p$ so that by 5.1(a) there exists a partition of unity $(e_j)_{j=1,\ldots,n}$ in P, with $\beta(e_j)=e_{j+1}$ $(j=1,\ldots,p)$ and:

$$\|[e_j, x_l]\|_2 \leqq \varepsilon_n \quad (j=1,\ldots,p;\ l=1,\ldots,n).$$

We then choose a system of matrix units $(f_r)_{r=1,\ldots,n^{2n}}$ in $(K_1\cup\ldots\cup K_n)''$ and write $x_j=\sum_r \lambda_{r,j} f_r y_{r,j}$ where the λ's are scalars, the y's belong to P_1, for $j=1,\ldots$ $\ldots, n+1$. Clearly we thus have a finite number k of elements of P_1, say $y_1,\ldots,y_k$, and an $\eta>0$ such that

$$(5.6)\qquad \begin{aligned} &\big(v \text{ unitary in } P,\ \|\beta(y_j)-vy_jv^*\|_2 \leqq \eta\ \ (j=1,\ldots,k)\big)\Rightarrow\\ &\Rightarrow\Big(\|\alpha(x_j)-\Big(\prod_1^n \operatorname{Ad} U_l\Big)\operatorname{Ad} v\,(x_j)\|_2\leqq\varepsilon_{n+1}\ \ (j=1,\ldots,n+1)\Big).\end{aligned}$$

We moreover assume that $\eta\leqq 2\varepsilon_n$.

We choose $\delta>0$ from the above $\eta>0$ and the lemma 4.3 with $\varepsilon=\frac{1}{4}\eta$. Now by corollary 3.3 we can find an element U of P, $\|U\|\leqq 1$, satisfying the following conditions:

$$(5.7)\qquad \|U^{p-l}-(U^*)^l\|_2\leqq\delta,\quad \|\beta(U)-U\|_2\leqq\delta,$$

$$\|U^p-1\|_2\leqq\delta,\quad \|Ue_iU^*-\beta(e_i)\|_2\leqq\delta\quad (i=1,\ldots,p)$$

and

$$\|Uy_jU^*-\beta(y_j)\|_2\leqq\frac{1}{2}\eta\quad (j=1,\ldots,k).$$

It then follows from lemma 4.3 applied to $\beta\in\operatorname{Aut} P$ that there exists a partition of unity $(E_j)_{j=1,\ldots,p}$ in P, a unitary $V\in P$ such that $VE_jV^*=E_{j+1}$ for all j, $V^p=1$, that the type I_p subfactor K of P generated by $(E_j)_{j=1,\ldots,p}$ and V is globally invariant under β, with $\beta/K=\operatorname{Ad} V/K$ and that

$$\|E_j-e_j\|_2\leqq\frac{1}{4}\eta,\quad \|V-U\|_2\leqq\frac{1}{4}\eta.$$

We shall take $K_{n+1}=K$, $U_{n+1}=V$. We have for $j\in\{1,\ldots,k\}$:

$$\|Vy_jV^*-\beta(y_j)\|\leqq 2\|V-U\|_2+\|Uy_jU^*-\beta(y_j)\|_2\leqq\eta$$

so that by 5.6 we get:

$$\Big\|\alpha(x_j)-\Big(\prod_1^{n+1}\operatorname{Ad} U_l\Big)(x_j)\Big\|_2\leqq\varepsilon_{n+1}\quad (j=1,\ldots,n+1).$$

But by the induction hypothesis we had:

$$\left\|\alpha(x_j)-\left(\prod_1^n \operatorname{Ad} U_l\right)(x_j)\right\|_2 \leqq \varepsilon_n \quad (j=1, \ldots, n).$$

This, using the fact that $\prod_1^n \operatorname{Ad} U_l$ preserves the $\| \ \|_2$ norm shows that $\|Vx_jV^*-$ $-x_j\|_2 \leqq \varepsilon_n+\varepsilon_{n+1} \leqq 2\varepsilon_n$ $(j=1, \ldots, n)$.

Also we have $\|[E_i, x_j]\|_2 \leqq \|[e_i, x_j]\|_2+2\|E_i-e_i\|_2 \leqq 2\varepsilon_n$ $(j=1, \ldots, n$ and $i=1, \ldots, p)$.

So it follows from 5.4 that:

$$\|E_{K'_{n+1}}(x_j)-x_j\|_2 \leqq \frac{1}{2^{n+1}} \quad (j=1, \ldots, n).$$

We have shown how to construct the sequence $(K_n)_{n\in\mathbf{N}}$ satisfying 5.2 and 5.5. Let $K=\left(\bigcup_{n\in\mathbf{N}} K_n\right)''$, then by (5.2) we have a splitting of R as a tensor product of K by K'_R. Let us note also by 5.5 that:

$$\alpha(x)=\left(\prod_1^\infty \operatorname{Ad} U_l\right)(x), \quad \forall x\in R.$$

This shows that α is conjugate to $s_p^1\otimes$(identity of K'_R); but α is conjugate to $\alpha\otimes 1_R$ (thm. 5.1(a)) and (Identity on K'_R)$\otimes$(Identity on R) is clearly 1_R because $K'_R\otimes R$ is isomorphic to R. So α is conjugate to $s_p^1\otimes 1_R$ which again by 5.1 (a) is conjugate to s_p^1. Q.E.D.

VI. The cyclic group of outer conjugacy classes with given outer period p

In this section we shall prove that for given $p\in\mathbf{N}$ and γ, $\gamma^p=1$, there is only one outer conjugacy class with outer invariants (p, γ). The proof relies on the study of the tensor product as a law of composition between outer conjugacy classes with outer period p.

Definition 6.1. Let R be the hyperfinite II_1 factor, a and b be outer conjugacy classes in $\operatorname{Aut} R$, then we let $a\times b$ be the outer conjugacy class of any automorphism $\alpha\otimes\beta\in\operatorname{Aut} R\otimes R$ with $\alpha\in a$, $\beta\in b$, brought back to $\operatorname{Aut} R$ by any isomorphism Π of R on $R\otimes R$.

In other words, for $\alpha\in a$, $\beta\in b$ and $\pi\colon R\to R\otimes R$ the automorphism $\pi^{-1}(\alpha\otimes\beta)\pi$ belongs to $a\times b$. Clearly changing α to $\alpha'\in a$, β to $\beta'\in b$, and π to $\pi'\colon R\to R\otimes R$ does not change the outer conjugacy class of $\pi^{-1}(\alpha\otimes\beta)\pi$, so that 6.1 makes sense.

Theorem 6.2. *For each* $p \in \mathbf{N}$ *the set* Br_p *of outer conjugacy classes in* $\operatorname{Aut} R$, *with outer period equal to* p, *endowed with the law of composition* $(a, b) \to a \times b$ *is an abelian group and* γ *is an isomorphism of this group on the group of* pth *roots of* 1 *in* $\mathbf{C}$.

Corollary 6.3. *For each* $p \in \mathbf{N}$, Br_p *is a cyclic group of order* p *with unit the outer conjugacy class of* s_p^1 *and generator the outer conjugacy class of* s_p^γ *if* γ *is a primitive* pth *root of* 1.

Proof. Immediate from 6.2 and proposition 1.6.

Corollary 6.4. *Let* R *be the hyperfinite* II_1 *factor,* $\alpha, \beta \in \operatorname{Aut} R$ *be periodic, then*

$$(\alpha \text{ conjugate to } \beta) \Leftrightarrow (p_0(\alpha) = p_0(\beta), \gamma(\alpha) = \gamma(\beta), \varepsilon(\alpha) = \varepsilon(\beta)).$$

Proof. By 6.2, α and β are outer conjugate iff they have the same outer invariants. By 2.8 if α and β are outer conjugate and $\varepsilon(\alpha) = \varepsilon(\beta)$ then α and β are conjugate. Q.E.D.

Proof of theorem 6.2. Let us first check that $a \in Br_p$, $b \in Br_p \Rightarrow a \times b \in Br_p$. By [9] Cor. 6 we know that the tensor product $\alpha \otimes \beta$ of two automorphisms α and β is inner if and only if both are inner. It follows in general that $p_0(\alpha \otimes \beta)$ equals l.c.m.$(p_0(\alpha), p_0(\beta))$ and in particular that Br_p is stable under $(a, b) \to a \times b$. Next we show that the class e of s_p^1 is a unit in Br_p. Let $a \in Br_p$ and let α be a minimal periodic automorphism. Then by theorem 5.1 (a) we know that $\alpha \otimes s_p^1$ is conjugate to α and hence that $a \times e$ is equal to a.

Let us now check that γ is an homomorphism; let $a, b \in Br_p$, $\alpha \in a$, $\beta \in b$, and $\alpha^p = \operatorname{Ad} u$, $\beta^p = \operatorname{Ad} v$, $u, v \in R$ with $\alpha(u) = \gamma(\alpha) u$, $\beta(v) = \gamma(\beta) v$. We then have $(\alpha \otimes \beta)^p = \operatorname{Ad}(u \otimes v)$ and $\alpha \otimes \beta(u \otimes v) = \alpha(u) \otimes \beta(v) = \gamma(\alpha)\gamma(\beta) u \otimes v$.

Let us prove that e is characterized in Br_p by the condition $\gamma(e) = 1$. Take $a \in Br_p$ with $\gamma(a) = 1$, and let α be minimal periodic. Then the period of α is equal to its outer period, equal to p. Hence by theorem 5.1 (b) α is conjugate to s_p^1, so that $a = e$.

We know therefore that Br_p is a group, we can in fact give a description of the inverse of an element a of Br_p: Let R^0 be the opposite von Neumann algebra of R, i.e., R^0 coincides with R as a complex vector space but the product is $(x, y) \to yx$ instead of $(x, y) \to xy$. Then let $\alpha \in \operatorname{Aut} R$, obviously α as a linear transformation of R defines an automorphism α^0 of R^0, which, because R^0 is hyperfinite and hence isomorphic to R, defines a conjugacy class in $\operatorname{Aut} R$, called the opposite of α. Clearly $p(\alpha) = p(\alpha^0)$. Let $\alpha^p = \operatorname{Ad} U$, $\alpha(U) = \gamma U$, then the equality $\alpha^p(x) = U x U^*$, $x \in R$, means that $(\alpha^0)^p(x) = U^* x U$, $x \in R^0$, so that, as $\alpha^0(U^*) = \alpha(U^*) = \bar{\gamma} U^*$, we get $\gamma(\alpha^0) = \gamma(\alpha)^{-1}$.

Of course a^0 is meaningful for $a \in Br_p$ and $a \times a^0$ is equal to e because $\gamma(a \times a^0) = 1$.

The end of the proof of 6.2 is now easy. We know that Br_p is a group, that γ is an homomorphism with trivial kernel and that γ is surjective by 1.6 (c). Q.E.D.

We now apply theorem 6.2 to determine the conditions under which two periodic automorphisms $\alpha, \beta \in \mathrm{Aut}\, R$ are weakly equivalent, i.e. there exists a $\sigma \in \mathrm{Aut}\, R$ such that $\sigma[\alpha]\sigma^{-1}=[\beta]$, where $[\alpha]$ is the full group, [4] p. 163, of the group $\{\alpha^n, n \in \mathbf{Z}\} \subset \subset \mathrm{Aut}\, R$.

Let $n=2^l m$, m odd, be an integer. Let S_n be the set consisting of all prime divisors of m with in addition an element ε if $l=2$ and two elements ε, ω if $l>2$. Let for each integer k prime relative to n, $\left(\frac{k}{n}\right) \in \{-1, 1\}^{S_n}$ be such that $\left(\frac{k}{n}\right)_\varepsilon = =(-1)^{\varepsilon(k)}$, $\varepsilon(k)=\frac{k-1}{2}$, $\left(\frac{k}{n}\right)_\omega =(-1)^{\omega(k)}$ where $\omega(k)=\frac{k^2-1}{8}$, and $\left(\frac{k}{n}\right)_p = \left(\frac{k}{p}\right)$ as in [11] p. 14 otherwise.

Theorem 6.5. *For a periodic* $\alpha \in \mathrm{Aut}\, R$ *define* $c(\alpha)=\mathrm{Order}\, \gamma(\alpha)$ *and* $q(\alpha)= =\left(\frac{k}{c(\alpha)}\right)$ *where* $\gamma(\alpha)=\exp(2\pi i k/c(\alpha))$.

(a) *Two periodic automorphisms* α *and* β *are weakly equivalent if and only if* $p_0(\alpha)=p_0(\beta)$, $c(\alpha)=c(\beta)$ *and* $q(\alpha)=q(\beta)$.

(b) *Let* c *and* d *be integers* $\geqq 1$, S_c *be defined as above, and* $q \in \{-1, 1\}^{S_c}$. *Then there exists a periodic* $\alpha \in \mathrm{Aut}\, R$ *such that* $p_0(\alpha)=cd$, $c(\alpha)=c$, $q(\alpha)=q$.

Proof. (a) If α is weakly equivalent to β then $p_0(\alpha)=p_0(\beta)=p$ because $p_0(\alpha)$ is the order of the image of $[\alpha]$ in $\mathrm{Out}\, R$. Also there exists an integer s, necessarily prime relative to p, and such that α is outer equivalent to β^s.

We have, with $p=p(\alpha)=p(\beta)$, that $\beta^p=\mathrm{Ad}\, U$, $\beta(U)=\gamma(\beta)U$ for some unitary $U \in R$. Hence $(\beta^s)^p=\mathrm{Ad}\, U^s$, $\beta^s(U^s)=\gamma(\beta)^{s^2}U$ so that $\gamma(\beta^s)=\gamma(\beta)^{s^2}$.

As s is prime relative to p it follows that the order of $\gamma(\beta)^{s^2}$ is the same as the order of $\gamma(\beta)$ so that $c(\alpha)=c(\beta)=c$. Put $\gamma(\alpha)=\exp\left(\frac{2\pi i k}{c}\right)$, $\gamma(\beta)=\exp\left(\frac{2\pi i k'}{c}\right)$. Then we have $\gamma(\alpha)=\gamma(\beta)^{s^2}$ so that in $\mathbf{Z}/c$ we get $k=s^2k'$, where k, k', s^2 are units in $\mathbf{Z}/c$. It hence follows that $q(\alpha)=q(\beta)$.

Conversely, assume that $p_0(\alpha)=p_0(\beta)$, $c(\alpha)=c(\beta)$, $q(\alpha)=q(\beta)$, and put $\gamma(\alpha)= =\exp\left(\frac{2\pi i k}{c}\right)$, $\gamma(\beta)=\exp\left(\frac{2\pi i k'}{c}\right)$. Then by [11] thm. 3, p. 39 and 1, p. 46, take s, prime relative to $p_0(\alpha)$, such that $ks^2=k'\,(c(\alpha))$. It follows that α^s has the same outer invariants as β and that α and β are weakly equivalent.

(b) Let $k \in \mathbf{Z}/c$ be a unit in $\mathbf{Z}/c$ such that $\left(\frac{k}{c}\right)=q$ ([11] lemma 1, p. 46) then take $\gamma=\exp\left(\frac{2\pi i k}{c}\right)$ and $\alpha=s_{cd}^{\gamma}$.

It follows that $p_0(\alpha)=cd$, $c(\alpha)=\text{Order }\gamma=c$, and $q(\alpha)=q$. Q.E.D.

Remark 6.7. By 6.5 there are automorphisms α of R, the simplest being s_3^j where $j^3=1, j\neq 1$, which are not weakly equivalent to their opposite α^0. One can deduce from this that the pair $R^\alpha\subset R$ is not isomorphic to the opposite pair.

Remark 6.8. Let $n\in \mathrm{N}$ and M be an arbitrary factor. Let $\alpha\in\operatorname{Aut}M$, $p_0(\alpha)=n$. Then consider the abstract kernel

$$q\in \mathbf{Z}/n \to \varepsilon(\alpha^q)\in \operatorname{Out} M$$

where ε is the canonical map from $\operatorname{Aut} M$ to $\operatorname{Out} M$. To this abstract kernel there corresponds an obstruction $k\in H^3(\mathbf{Z}/n, \mathbf{T})$ ([12] p. 216) with $\mathbf{T}$=center of the unitary group of M (i.e. $\mathbf{T}=\{z\in \mathrm{C}, |z|=1\}$) with trivial action of $\mathbf{Z}/n$. To get k one takes for $q\in\mathbf{Z}/n$ an arbitrary $\beta_q\in\operatorname{Aut} M$ with $\varepsilon(\beta_q)=\varepsilon(\alpha^q)$, then one takes for $q_1, q_2\in\mathbf{Z}/n$ a unitary u_{q_1,q_2} of M with $\operatorname{Ad} u_{q_1,q_2}=\beta_{q_1}\beta_{q_2}\beta^{-1}_{q_1+q_2}$ and: $k(q_1, q_2, q_3)=\beta_{q_1}(u_{q_2,q_3})u_{q_1,q_2+q_3}u^{-1}_{q_1+q_2,q_3}u^{-1}_{q_1,q_2}\in \mathbf{T}$. With the choice $\beta_q=\alpha^q$, one gets:

$$k(q_1, q_2, q_3) = \gamma(\alpha)^{q_1\eta(q_2,q_3)} \quad\text{where}\quad \eta(q_2, q_3) = \begin{cases} 0 & \text{if} \quad q_2+q_3 < n,\\ 1 & \text{if} \quad q_2+q_3 \geqq n.\end{cases}$$

Comparing the bar resolution of the trivial $\mathbf{Z}/n$ module $\mathbf{Z}$ with its periodic resolution of period 2 one brings back k to the element $\gamma(\alpha)$ of $\{a\in\mathbf{T}, a^n=1\}$.

VII. Applications to various questions of noncommutative ergodic theory

Throughout we let R be the hyperfinite factor of type II_1. This section is devoted to apply theorem 1.5 to answer the following questions.

Problem 7.1. Is any periodic $\alpha\in\operatorname{Aut} R$ conjugate to the opposite of its inverse? (It is easy to show that there are inner automorphisms which are neither conjugate to their opposite nor their inverse. However they are always conjugate to the opposite of their inverse, when they are periodic).

Theorem 7.2. *Let* $\alpha\in\operatorname{Aut} R$ *be periodic. Then* α *is conjugate to* $(\alpha^0)^{-1}$ *if and only if* $\gamma(\alpha)^2=1$.

Proof. We have $\gamma(\alpha^{-1})=\gamma(\alpha)$ $(\alpha^{-p}=\operatorname{Ad} U^*$ where $p=p_0(\alpha)$ and $\alpha(U)=$ $=\gamma(\alpha)U$, so that $\alpha^{-1}(U^*)=\gamma(\alpha)U^*)$.

Hence, $\gamma((\alpha^{-1})^0)=\overline{\gamma(\alpha)}$ so that if $\gamma(\alpha)^2\neq 1$, α is not even *outer* conjugate to $(\alpha^0)^{-1}$.

We have $\varepsilon(\alpha^0)=\varepsilon(\alpha^{-1})$ for any $\alpha\in\operatorname{Int} R$, α periodic. Hence $\varepsilon(\alpha^0)=\varepsilon(\alpha^{-1})$ holds for any periodic $\alpha\in\operatorname{Aut} R$ (because $(\alpha^0)^{p_m}=(\alpha^{p_m})^0$, $(\alpha^{-1})^{p_m}=(\alpha^{p_m})^{-1}$).

So 7.2 follows from 1.5 and the equalities:

$$p_0((\alpha^0)^{-1}) = p_0(\alpha), \quad \gamma((\alpha^0)^{-1}) = \overline{\gamma(\alpha)}, \quad \varepsilon((\alpha^0)^{-1}) = \varepsilon(\alpha).$$

In particular if $\gamma(\alpha)^2 \neq 1$, α cannot be outer conjugate to an infinite tensor product of automorphisms of finite dimensional factors, because such automorphisms are conjugate to the opposite of their inverse. This drives to:

Problem 7.3. Which automorphisms $\alpha \in \operatorname{Aut} R$, α periodic, are conjugate (resp. outer conjugate) to an infinite tensor product of automorphisms of finite dimensional factors? To infinite tensor product of inner automorphisms of arbitrary factors?

Theorem 7.4. *Let* $\alpha \in \operatorname{Aut} R$, α *periodic, then:*

(a) *If* α *is an infinite tensor product of inner automorphisms* $\operatorname{Ad} U_j$ *of finite factors* R_j, *then* $\gamma(\alpha) = 1$.[10]

(b) *If* $\gamma(\alpha) = 1$, α *is the tensor product of an inner automorphism of* R *by an infinite tensor product of automorphisms of finite dimensional factors.*

(c) *Let* α *be periodic of period* p, *with* $\gamma(\alpha) = 1$. *Put* $p = qp_0(\alpha)$, *assume* q *prime, let* $\varepsilon = \sum_{j=0}^{q-1} \lambda_j \, \varepsilon(e^{i2\pi j/n})$ *be the inner invariant of* α. *Then* α *is an infinite tensor product of automorphisms of finite dimensional factors if and only if either all the* λ_j' *are rational numbers or they are all* $\neq 0$ *as well as the* $\hat{\lambda}_k$:

$$\hat{\lambda}_k = \Sigma \lambda_j \exp \frac{i2\pi jk}{n} \quad k = 0, \ldots, q-1.$$

Proof. Can be left to the reader.

One has the following positive general result concerning the approximation of periodic automorphisms of R by automorphisms of finite dimensional von Neumann algebras:

Theorem 7.5. *Let* α *be a periodic automorphism of* R, *then there exists an increasing sequence of finite dimensional subalgebras* P_n *of* R *such that* $\alpha(P_n) = P_n$ *for all* n *and that* $\bigcup_{n=1}^{\infty} P_n$ *is strongly dense in* R.

Proof. First take $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$ and s_p^γ as constructed in part 1. Consider $P_n = (F_p^{(1,n)} \cup \{\theta^n(U_\gamma)\})''$ with the notations of proposition 1.6. Then, as $\alpha\theta^n(U_\gamma) = \gamma\theta^n(U_\gamma)$ with $\alpha = s_p^\gamma$ and as $\alpha(F_p^{(1,n)})$ is contained in the algebra generated by $F_p^{(1,n)}$ and $\theta^{n-1}(v_\gamma)$, i.e., by $F_p^{(1,n)}$ and $\theta^n(U_\gamma^*)$, we see that P_n is globally invariant under α for each n.

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[10]) Hence if $\gamma(\alpha) \neq 1$, α is not outer conjugate to such a tensor product.

Having proven 7.5 for the s_p^γ's we just have to prove it for periodic inner automorphisms and conclude using 1.11.

Let $\operatorname{Ad} U$ be a periodic inner automorphism of R, with $U=\sum_{j=1}^{m} a_j e_j$ where $a_j \in \mathbf{C}$, $a_j^m=1$ and the e_j's are projections in R.

Choose an increasing sequence of projections $f_n \in R$ commuting with U, such that for each n, j: $\tau(f_n e_j)$ is a dyadic rational, and with $f_n \to 1$ when $n \to \infty$.

Let $(x_k)_{k\in\mathbf{N}}$ be a dense sequence in the unit ball of R (dense for the strong topology). Then by induction on n one builds a sequence $(K_n)_{n\in\mathbf{N}}$ where K_n is a subfactor of type $\mathrm{I}_{2^{p_n}}$ of R_{f_n}, containing Uf_n and such that $K_{n-1}+\mathbf{C}(f_n-f_{n-1})\subset \subset K_n$ and that it approximates the $f_n x_j f_n$ $(j=1, \dots, n)$ up to $\frac{1}{n}$ in the trace norm.

It follows that the sequence $P_n=K_n+\mathbf{C}(1-f_n)$ is increasing, that it generates R and that $UP_nU^*=P_n$ for each $n\in\mathbf{N}$. Q.E.D.

Problem 7.6. Which periodic automorphisms of R have a square root?

Clearly, any inner automorphism of R has a square root, and hence by 1.11 we see that a periodic α with outer invariants (p, γ) has a square root in $\operatorname{Aut} R$ if s_p^γ has one. To compute $p_0(\alpha^2)$, $\gamma(\alpha^2)$ we distinguish two cases:

(1) $p_0(\alpha)$ is odd. Then α^{2q} is outer for $q=1, \dots, p_0(\alpha)-1$ because $2q$ is not a multiple of $p_0(\alpha)$. So:

$$p_0(\alpha^2) = p_0(\alpha), \quad \gamma(\alpha^2) = \gamma(\alpha)^4.$$

(2) $p_0(\alpha)$ is even. Then $(\alpha^2)^{\frac{p_0(\alpha)}{2}}$ is inner, and

$$p_0(\alpha^2) = \frac{1}{2} p_0(\alpha), \quad \gamma(\alpha^2) = \gamma(\alpha)^2.$$

Theorem 7.7. [11]) *Let $p\in\mathbf{N}$, $\gamma\in\mathbf{C}$, $\gamma^p=1$. If the order of γ is odd, then any periodic automorphism with outer invariant (p, γ) has a square root.*

If the order of γ is even then s_p^γ has no square root.

Proof. Assume $\operatorname{Order}\gamma=2q+1$. Put $\gamma'=\gamma^{-q}$, then γ' is a root of 1 of an order dividing the order of γ and $\gamma'^2=\gamma^{-2q}=\gamma$. So $s_{2p}^{\gamma'}$ has a period dividing $2\cdot$ period of s_p^γ. As $2p$ is even we have $p_0((s_{2p}^{\gamma'})^2)=p$, $\gamma((s_{2p}^{\gamma'})^2)=\gamma'^2=\gamma$, and

$$(s_{2p}^{\gamma'})^{2p\cdot\operatorname{Order}\gamma} = 1.$$

So $(s_{2p}^{\gamma'})^2$ has outer invariants (p, γ) and trivial inner invariant, so that it is con-

[11]) Clearly any $\alpha\in\operatorname{Aut} R$ with odd period, say $2m+1$, has a square root, namely α^{m+1}.

jugate to s_p^γ by theorem 1.5. Hence $s_p^\gamma \otimes \mathrm{Ad}\, U$ has a square root for all inner automorphisms and theorem 1.11 applies.

If the square roots γ', γ'' of γ satisfy Order γ'=Order $\gamma''=2$ order γ, take α such that $\alpha^2=s_p^\gamma$. Then we must have $p_0(\alpha)=2p_0(\alpha^2)$ because $p_0(\alpha)$ must be even. Then also $\gamma(\alpha)^2=\gamma$, so that, say, $\gamma(\alpha)=\gamma'$. We have:

$$(\text{period } \alpha) \text{ is a multiple of } (\text{period } s_{2p}^{\gamma'}).$$

Hence, as (period $s_{2p}^{\gamma'})=2p\cdot$Order $\gamma'=4p\cdot$Order γ we see that we cannot have $\alpha^{2p\,\mathrm{Order}\,\gamma}=1$ as required by $\alpha^2=s_p^\gamma$. Q.E.D.

Remark 7.8. In Out R any periodic element has a qth root for any $q\in\mathbf{N}$, $q\neq 0$, because $(s_p^{\gamma q})^q$ is outer conjugate to s_{pq}^γ for all p, q and $\gamma, (\gamma^q)^p=1$.

Remark 7.9. In [2] H. Borchers studies automorphisms α of von Neumann algebras M and their relations with inner automorphisms. For each $n\in\mathbf{N}$ he introduces a class K_n of automorphisms, and theorem 4.1 of [2] states that, when M is a factor for simplicity, $(\alpha^i$ is inner iff $i=0\,(n))\Leftrightarrow \alpha\in K_n$.

However the automorphisms s_p^γ, $\gamma\neq 1$, give a counterexample to this theorem because by [2] prop. 4.7, if $\alpha\in K_n$ then α^n is of the form Ad U with $U\in M^\alpha$. (In the notations of [2] $U\in Z_0$ where (Def. 2.1) Z_0 denotes the center of the fixed point algebra.) However if in [2] one replaces everywhere the word "inner" by "inner implemented in Z_0" then all the argument goes through.

Remark 7.10. In [8] thm. 1, V. Ya. Golodets claims that the cross product of the hyperfinite II_1 factor R by any cyclic group G of outer automorphisms is again hyperfinite. This theorem is true from our above results. (Apply 7.5.) However the proof given in [8] does not work. To see this we take the notations of [8]. The automorphism h of $\mathcal{M}=G\times M$ corresponds to the dual action of Takesaki, of the generator of G associated to ε (ε is a primitive nth root of 1). Hence in $G_h\times\mathcal{M}$ the commutant of the type I_n factor generated by V_g and V_h is, by the duality, the von Neumann algebra $\tilde{\Pi}(M)$, where $\tilde{\Pi}$ is an isomorphism of M into $G_h\times\mathcal{M}$ defined by

$$\tilde{\Pi}(x)=\Sigma\hat{x}_q V_h^q,\quad x=\Sigma\hat{x}_q,\quad g(\hat{x}_q)=\varepsilon^q\hat{x}_q.$$

Now $\mathcal{M}$, as a subfactor of $G_h\times\mathcal{M}$, has $\mathbf{C}$ as relative commutant so that the normalizer of $\mathcal{M}$ in $G_h\times\mathcal{M}$ consists only of unitaries of the form vV_h^m, v unitary in $\mathcal{M}$, $0\leqq m<n$.

We can hence find a unitary X in $G_h\times\mathcal{M}$ which commutes with V_g and V_h, but for which $X\mathcal{M}X^*\neq\mathcal{M}$.

The claim in [8] is that for any automorphism φ of $G_h\times\mathcal{M}$ for which $\varphi(W_1)=$ $=V_h$, $\varphi(W_2)=V_g$, the family of operators $\varphi(W_2)=V_g$, $V_k=\varphi(W_k)$, $k=3, 4, \ldots$ generates $\mathcal{M}$.

But if this is true for some φ, replace φ by $\varphi' = (\mathrm{Ad}\, X)\varphi$ with X as above, then certainly $\varphi'(W_1) = V_h$, $\varphi'(W_2) = V_g$, but the $\varphi'(W_p)$ $(p = 2, 3, 4, \ldots)$ generate $X\mathcal{M}X^*$ which is different from $\mathcal{M}$, so that the condition would fail for φ'.

Bibliography

[1] Arveson, On group of automorphisms of operator algebras, *J. Funct. Analysis*, **15** (1974), 217—243.

[2] H. Borchers, Characterization of inner * automorphisms of W^* algebras, *Publ. R.I.M.S. Kyoto Univ.*, **10** (1974), 11—50.

[3] E. Christensen, Perturbation of operator algebras, *I.S.B.N. Oslo Preprint Series* No. 14, July 74.

[4] A. Connes, Une classification des facteurs de type. III, *Ann. Sci. E.N.S.*, (4) **6** (1973), 133—252.

[5] A. Connes, Almost periodic states and factors of type III_1, *J. Funct. Analysis* **16** (1974), 415—445.

[6] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, 2ed., Gauthier Villars (Paris, 1969).

[7] D. McDuff, Central sequences and the hyperfinite factor, *Proc. London Math. Soc.*, **21** (1970), 443—461.

[8] V. Ya Golodets, On von Neumann's approximately finite algebras with finite trace, *Soviet Math. Dokl.*, **9** (1968), No. 4.

[9] R. Kallman, A decomposition theorem for automorphisms of von Neumann algebras, *Funct. Analysis*, Edited by C. O. Wilde, Academic Press (1970), pp. 33—35.

[10] S. Sakai, *C^* and W^* algebras* (Ergebnisse der Mathematik und ihrer Grenzgebiete Bd. 60), Springer-Verlag (Berlin—Heidelberg—New York, 1971).

[11] J. P. Serre, *Cours d'arithmétique*, Presse Universitaire de France.

[12] Z. Takeda, On the extensions of finite factors. II, *Proc. Japan Acad.*, **35** (1959), 215—220.

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